(8+8+8 pls) 1. Evaluate each of the following limits if it exists.

(a) \( \lim_{x \to -\infty} \left( x + \sqrt{x^2 - x - 4} \right) = \lim_{x \to -\infty} \frac{\left( x + \sqrt{x^2 - x - 4} \right) \left( x - \sqrt{x^2 - x - 4} \right)}{x - \sqrt{x^2 - x - 4}} \)

\[ = \lim_{x \to -\infty} \frac{x + 4}{x - \sqrt{x^2 - x - 4}} \]

\[ = \lim_{x \to -\infty} \frac{x + 4}{x + x \sqrt{x^2 - x - 4}} \]

\[ = \lim_{x \to -\infty} \frac{1 + \frac{4}{x}}{1 + \frac{\sqrt{x^2 - x - 4}}{x}} \]

\[ = \frac{1}{2} \]

(b) \( \lim_{x \to 0} x \sin(x) = 0 \)

Note that \( \lim_{x \to 0} x = 0 \) and \( \lim_{x \to 0} \sin(x) = 0 \), and hence

\[ \lim_{x \to 0} (x \sin(x)) = \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin(x) = 0 \]

Both limits exist.

(c) \( \lim_{x \to 0} \frac{x^2 \sin \left( \frac{1}{x} \right)}{\sin(x)} \)

\[ = \lim_{x \to 0} \frac{x}{\sin(x)} \cdot \frac{\sin \left( \frac{1}{x} \right)}{\frac{1}{x}} \]

\[ = 1 \cdot 0 = 0 \]

Since \( \lim_{x \to 0} \frac{x}{\sin(x)} = 1 \) and \( \lim_{x \to 0} \frac{\sin(1/x)}{1/x} = 0 \),

\[ \frac{x^2}{\sin(x)} \cdot \frac{\sin \left( \frac{1}{x} \right)}{\sin(x)} \leq \frac{x^2 \sin \left( \frac{1}{x} \right)}{\sin(x)} \leq \frac{x^2}{\sin(x)} \]

\[ \Rightarrow \lim_{x \to 0} \frac{x^2 \sin \left( \frac{1}{x} \right)}{\sin(x)} = 0 \]

0 as \( x \to 0 \)
(12 pts) 2. Find an equation of the normal line to the curve \( x^3 + y^3 - 9xy = 0 \) at the point \((4, 2)\).

By implicit differentiation, we have

\[
3x^2 + 3y^2 y' - 9y - 9xy' = 0 \implies y' = \frac{3y - x^2}{y^2 - 3x}
\]

whenever it exists.

Then,

\[
y'(4, 2) = \frac{3(2) - 16}{4 - 12} = \frac{-10}{-8} = \frac{5}{4}
\]

a slope of tangent line through the point \((4, 2)\).

So, the slope of the normal line is \(-\frac{4}{5}\).

So, the eqn is

\[
y - 2 = -\frac{4}{5} (x - 4)
\]

(12 pts) 3. A point is moving to the right along the first-quadrant portion of the curve \(x^2y^3 = 72\). When the point has coordinates \((3, 2)\), its horizontal velocity is 2 units/second. What is the rate of change in the distance of the particle to the origin?

Let \(x = x(t)\) and \(y = y(t)\) be coordinate functions parameterized by time \(t\), so that \(x^2y^3 = 72\) and \(\frac{dx}{dt} \big|_{(3,2)} = 2\) units/sec.

Let \(D(t) = \sqrt{x^2 + y^2}\) be the distance of the particle to the origin. \(D > 0\).

We need to find \(\frac{dD}{dt} \big|_{(3,2)}\).

Differentiate \(x^2y^3 = 72\) wrt \(t\), then we have

\[
2xy^3\frac{dx}{dt} + 3x^2y^2\frac{dy}{dt} = 0 \implies \frac{dy}{dt} \bigg|_{(3,2)} = -\frac{8}{9}
\]

Differentiate \(D^2 = x^2 + y^2\) wrt \(t\), then we have

\[
2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \implies \frac{dD}{dt} \bigg|_{(3,2)} = D \frac{dD}{dt} \bigg|_{(3,2)} = 6 + \frac{-16}{9}
\]

So,

\[
\frac{dD}{dt} \bigg|_{(3,2)} = \frac{38}{9\sqrt{13}}
\]
4. Let \( f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ \cos(x) & \text{if } x > 0. \end{cases} \)

Is \( f \) continuous at \( x = 0 \)? Explain why or why not. If \( f'(0) \) exists, determine its value; if not, explain why it does not exist.

1. Need to check \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \). 
   \[
   \lim_{x \to 0^-} \frac{\sin(x)}{x} = 1 = \lim_{x \to 0^+} \cos(x) = f(0)
   \]
   \( \Rightarrow \) \( f \) is cont. at \( x = 0 \).

2. Now, we need to check \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) \) exists or not.
   
   \[
   \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\sin(x) - 1}{x} = \lim_{x \to 0^-} \frac{\sin(x) - x}{x^2} = \lim_{x \to 0^-} \frac{\sin(x) - x}{2x} = \lim_{x \to 0^-} \frac{-\sin(x)}{2x^2} = \lim_{x \to 0^-} \frac{-\sin(x)}{2} = 0
   \]
   
   \[
   \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{\cos(x) - 1}{x} = \lim_{x \to 0^+} \frac{-\sin(x)}{x} = -1
   \]
   
   So, \( f'(0) \) exists and \( f'(0) = 0 \).

5. Prove that the function \( f(x) = x^3 + 5x + 2 \) has a unique zero in \( \mathbb{R} \).

Clearly \( f \) is cont. everywhere since it is polynomial.

In particular, \( f \) is also cont. on any interval \( I = [a, b] \subset \mathbb{R} \).

Take \( I = [-1, 0] \), then we also have \( f(0) = 2 > 0 \), \( f(-1) = -4 < 0 \).

By IVT, there is some \( c \in [-1, 0] \) so \( f(c) = 0 \).

So, \( f \) is at least one zero. \( \Box \) (take \( c \in (-1, 0) \))

Now, assume that \( f \) has more than one root, say \( c_1 \) and \( c_2 \).

Notice that \( f \) is clearly cont. on \( [c_1, c_2] \) and differentiable on \( (c_1, c_2) \) since it is polynomial. By Rolle’s thm, there exists some \( d \in (c_1, c_2) \) such that \( f'(d) = 0 \).

But, \( f'(x) = 3x^2 + 5 \) which has no real root. Therefore, assumption is false. i.e., \( f \) cannot have more than one root. \( \Box \)

By (1) and (2), it means \( f \) has exactly one zero.
(9+9+9 pts) 6.
(a) Find \( \frac{dy}{dx} \) if \( x^y = y^x \)

Take the logarithm of both sides: \( \ln x^y = \ln y^x \) \( \Rightarrow \) \( y \ln x = x \ln y \) \( \text{(A)} \)

Take the derivative of both sides of \( \text{(A)} \) wrt \( x \), then

\[
y' \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y} \cdot y'
\]

\( \Rightarrow \)

\[
y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}
\]

(b) Find \( \frac{dy}{dx} \) if \( y = \sinh(x)^{\ln x} \)

Take the log. of both sides: \( \ln y = \ln(\sinh(x)^{\ln x}) \)

Take the derivative of both sides wrt \( x \), then we have

\[
\frac{y'}{y} = \frac{1}{x} \ln(\sinh(x)) + \ln x \cdot \frac{\cosh(x)}{\sinh(x)}
\]

\( \Rightarrow \)

\[
y' = \left[ \sinh(x) \right]^{\ln x} \left[ \frac{\ln(\sinh(x))}{x} + \ln x \cdot \frac{\cosh(x)}{\sinh(x)} \right]
\]

(c) Find \( \frac{d}{dx} f^{-1}(1) \) where \( f(x) = x^5 + 6x^3 + x + 1 \).

Let \( f^{-1}(1) = a \), then we have \( f(a) = 1 \)

i.e. \( a^5 + 6a^3 + a + 1 = 1 \) \( \Rightarrow \) \( a = 0 \)

 Recall \( (f^{-1})'(1) = \) \( \frac{1}{f'(f^{-1}(1))} \), so \( (f^{-1})'(1) = \frac{1}{f'(0)} \)

Since \( f'(x) = 5x^4 + 18x^2 + 1 \) and \( f'(0) = 1 \), we have

\[
(f^{-1})'(1) = 1
\]