M E T U
Department of Mathematics

	Calculus with Analytic Geometry First Midterm Exam	
Code : MATH 119 Acad.Year : 2017 Semester : Fall Coord. : Fırat Arıkan Date : 18.11.2017 Time : 9.30 Duration : 120 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 6 Questions on 4 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5	Q6	SHOW YOUR WORK !
SOLUTIONS						

Muhiddin UGUZ

Q.1 (4 × 5 = 20 pts) Consider the function $y = f(x) = e^{\arctan\left(\sqrt{1+\ln x}\right)}$.

(a) Find the **domain** of f. In order for $f(x)$ to be defined, we need to have $x>0$, $1+\ln x \geq 0$ and so $x \geq \frac{1}{e}$. For any $x \geq \frac{1}{e}$, $f(x)$ is defined and hence

$$\mathrm{dom}(f) = [\tfrac{1}{e}, +\infty)$$

(b) Find $f'(x)$.

$$f'(x) = e^{\arctan(\sqrt{1+\ln x})} \cdot \left(\arctan(\sqrt{1+\ln x})\right)'$$

$$= e^{\arctan(\sqrt{1+\ln x})} \cdot \frac{1}{1+(1+\ln x)} \cdot \left(\sqrt{1+\ln x}\right)'$$

$$= e^{\arctan(\sqrt{1+\ln x})} \cdot \frac{1}{2+\ln x} \cdot \frac{1}{2\sqrt{1+\ln x}} \cdot \frac{1}{x}$$

(c) Show that the function is **invertible** on its domain.

For $x > \frac{1}{e}$, each of $e^{\arctan(\sqrt{1+\ln x})}$, $\frac{1}{2+\ln x}$, $\frac{1}{2\sqrt{1+\ln x}}$ and $\frac{1}{x}$ is positive and hence $f'(x) > 0$. Also f is right-cont. at $x = \frac{1}{e}$. Thus, f is $\nearrow$ on $[\frac{1}{e}, \infty)$. It follows that f is one-to-one and hence invertible on its domain.

(d) Let $f^{-1}(x)$ be the inverse function. Find its derivative $\dfrac{d}{dx}f^{-1}(x)$ at $x = e^{\pi/4}$.

We have $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$ and $f(1) = e^{\frac{\pi}{4}}$

Hence $$(f^{-1})'(e^{\frac{\pi}{4}}) = \frac{1}{f'(f^{-1}(e^{\frac{\pi}{4}}))} = \frac{1}{f'(1)}$$

$$= \frac{1}{e^{\frac{\pi}{4}} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{1}} = \frac{4}{e^{\pi/4}}$$

SOLUTIONS

Q.2 ($4 \times 5 = 20$ pts) **Without using L'Hospital's rule**, evaluate the following limits or show they do NOT exist.

(a) $\lim\limits_{x\to\infty} \frac{2\sin(x^5)}{x^4+1}$ We have $-1 \leq \sin(x^5) \leq 1$ and so

$$\frac{-2}{x^4+1} \leq \frac{2\sin(x^5)}{x^4+1} \leq \frac{2}{x^4+1} \text{ for all } x.$$

Moreover, $\lim\limits_{x\to\infty} \frac{-2}{x^4+1} = \lim\limits_{x\to\infty} \frac{2}{x^4+1} = 0$ and hence, by the squeeze theorem, we have

$$\lim_{x\to\infty} \frac{2\sin(x^5)}{x^4+1} = 0$$

(b) $\lim\limits_{x\to\infty} [\ln(x^2+1) - \ln(x^2-1)] = \lim\limits_{x\to\infty} \ln\left(\frac{x^2+1}{x^2-1}\right)$ by properties of ln.

Moreover, $\lim\limits_{x\to\infty} \frac{x^2+1}{x^2-1} = \lim\limits_{x\to\infty} \frac{1+\overbrace{\frac{1}{x^2}}^{\to 0}}{1-\underbrace{\frac{1}{x^2}}_{\to 0}} = 1$. Since $\ln(x)$ is continuous at $x=1$, we have

$$\lim_{x\to\infty} \ln\left(\frac{x^2+1}{x^2-1}\right) = \ln\left(\lim_{x\to\infty} \frac{x^2+1}{x^2-1}\right) = \ln(1) = 0$$

(c) $\lim\limits_{x\to 2} \frac{x^2-4}{3x|x-2|}$ We have $\lim\limits_{x\to 2^+} \frac{x^2-4}{3x|x-2|} = \lim\limits_{x\to 2^+} \frac{(x-2)(x+2)}{3x(x-2)}$

$= \lim\limits_{x\to 2^+} \frac{x+2}{3x} = \frac{4}{6}$ and $\lim\limits_{x\to 2^-} \frac{x^2-4}{3x|x-2|} = \lim\limits_{x\to 2^-} \frac{(x-2)(x+2)}{3x\cdot -(x-2)} = \lim\limits_{x\to 2^-} -\frac{x+2}{3x} = \frac{-4}{6}$

Since right and left limits at $x=2$ are different, $\lim\limits_{x\to 2} \frac{x^2-4}{3x|x-2|}$ does not exist.

(d) $\lim\limits_{x\to 0} \frac{\cos x - e^x}{x} = \lim\limits_{x\to 0} \frac{(\cos x - e^x) - (\cos 0 - e^0)}{x} = f'(0)$

where $f(x) = \cos x - e^x$. Thus, since $f'(x) = -\sin x - e^x$,

$$\lim_{x\to 0} \frac{\cos x - e^x}{x} = f'(0) = -\sin 0 - e^0 = -1$$

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SOLUTIONS

Q.3 (10 pts) Find the number $a \neq 0$ such that the function

$$f(x) = \begin{cases} \dfrac{\sin(ax)}{x} + \cos x & \text{for } x < 0 \\ x^2 + 3e^{2x} & \text{for } x \geq 0 \end{cases}$$

is continuous **everywhere**.

For $x_0 > 0$, $f(x)$ is continuous at $x = x_0$ since $\frac{\sin(ax)}{x} + \cos x$ is continuous at $x = x_0$. Similarly, for $x_0 < 0$, $f(x)$ is continuous at $x = x_0$ since $x^2 + 3e^{2x}$ is continuous at $x = x_0$. Thus we need to find some $a \neq 0$ which makes f continuous at $x = 0$. f is continuous at $x = 0$ if and only if we have $\lim_{x\to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$. Therefore, f is continuous everywhere if and only if we have

$$f(0) = 3 = \lim_{x\to 0^+}(x^2 + 3e^{2x}) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin(ax)}{x} + \cos x\right) = \lim_{x\to 0^-}\left(\frac{\sin(ax)}{ax}\cdot a + \cos x\right) = a + 1$$

Thus, for $a = 2$, f is continuous everywhere

Q.4 (4 × 5 = 20 pts)

(a) By definition, the limit $\displaystyle\lim_{h\to 0} \frac{\cos h - e^{h^2} + h^5}{h}$ represents the derivative of a function $f(x)$ at the number $x = x_0$. Find $f(x)$ and x_0.

$$\lim_{h\to 0} \frac{\cosh - e^{h^2} + h^5}{h} = \lim_{h\to 0} \frac{\cosh - e^{h^2} + h^5 - 0}{h} = f'(x_0) \text{ where}$$

$f(x) = \cos x - e^{x^2} + x^5$ and $x_0 = 0$

(b) Calculate y' at the point $(x, y) = (1, -1)$ if $y = y(x)$ and $x^4 + y^4 + 2xy = 0$.

Using implicit differentiation, we get $4x^3 + 4y^3 \cdot y' + (2y + 2x \cdot y') = 0$. Plugging in $(x,y) = (1,-1)$ gives $4 - 4 \cdot y'\big|_{(1,-1)} + (-2 + 2y'\big|_{(1,-1)}) = 0$ and hence

$$y'\Big|_{(1,-1)} = \frac{2}{2} = 1$$

(c) Find $f'(x)$ if $f(x) = (\cos x + 3)^{\sqrt{2 + \sin x}}$.

$$f(x) = e^{\ln\left((\cos x+3)^{\sqrt{2+\sin x}}\right)} = e^{\sqrt{2+\sin x}\,\ln(\cos x+3)} \text{ and hence}$$

$$f'(x) = e^{\sqrt{2+\sin x}\,\ln(\cos x + 3)}\left[\left(\frac{1}{2\sqrt{2+\sin x}} \cdot \cos x\right) \cdot \ln(\cos x + 3) + \sqrt{2 + \sin x} \cdot \frac{1}{\cos x + 3} \cdot (-\sin x)\right]$$

(d) Find $f'(x)$ if $f(x) = 1 + \arcsin(x^2) + 3^x + \log_2 x$ ·

$$f'(x) = 0 + \frac{2x}{\sqrt{1 - x^4}} + 3^x \cdot \ln 3 + \frac{1}{x}\,\frac{1}{\ln 2}$$

SOLUTIONS

Q.5 **(5 + 10 + 5 = 20 pts)** Consider the function $f(x) = e^x + 2x + x^4$.
(a) Find the line **tangent** to the graph of $y = f(x)$ at the point $P(0, 1)$.

$f'(x) = e^x + 2 + 4x^3$ and hence the slope of the tangent line at point $(0,1)$ is $f'(0) = 3$. Therefore the equation of this line is $y - 1 = 3(x - 0)$

(b) Show that there exists a point $Q(x, y)$ on the graph of $y = f(x)$ at which the **tangent** line is **parallel** to the line $y = 2x + 1$. (Do NOT find the point Q.)

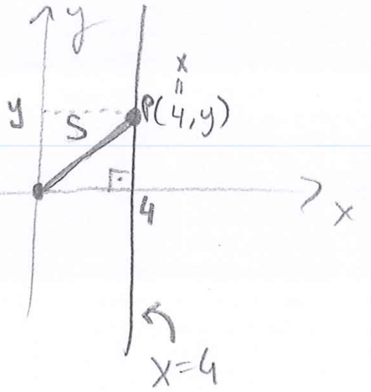
At such a point, slopes must be equal.
Therefore, $f'(x) = 2$. Note that $f'(x) = e^x + 2 + 4x^3$ is continuous everywhere and $f'(-1) = \frac{1}{e} + 2 - 4 < 2$ and $f'(0) = 3 > 2$. Therefore, applying IVT to $f'(x)$ on the interval $[-1, 0]$, we get that there exists $-1 < x < 0$ such that $f'(x) = 2$.

(c) In **Part (b)**, show that Q is unique, i.e. there is ONLY ONE such point.

$f''(x) = e^x + 12x^2 > 0$ for all x. Thus f' is increasing and hence one-to-one. Consequently, there can be only one point at which we can have $f'(x) = 2$

OR If there were $c_1 \neq c_2$ s.t. $f'(c_1) = f'(c_2) = 2$, then by Rolle's Thm $\exists d \in (c_1, c_2)$ s.t. $f''(d) = 0$. But $f''(x) = e^x + 12x^2 > 0$ $\forall x \in \mathbb{R}$, contradiction!

Q.6 **(10 pts)** A particle moves along the vertical line $x = 4$ with its y-coordinate increasing at a rate of 2 cm/sec. How fast is the **distance**, say S, of the particle to the **origin** changing when the particle is at the point $(4, 3)$?



By Pythagorean thm, $16 + y^2 = S^2$ and hence $2y \frac{dy}{dt} = 2S \frac{dS}{dt}$ by differentiating both sides

we are given that $\frac{dy}{dt} = 2$ cm/sec.

Thus, at the time when particle passes through $(4,3)$, we have

$$\cancel{2} \cdot 3 \cdot 2 = \cancel{2} \sqrt{16 + 3^2} \frac{dS}{dt}$$

So, $\frac{dS}{dt} = 6/5$ cm/sec.

OR $S = \sqrt{16 + y^2}$ and hence $\frac{dS}{dt} = \frac{\cancel{2}y}{\cancel{2}\sqrt{16 + y^2}} \cdot \frac{dy}{dt}$.

At $y = 3$, $\frac{dS}{dt} = \frac{3}{\sqrt{16+9}} \cdot 2 = 6/5$ cm/sec

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY Midterm 2	
Code : *Math 119* Acad. Year : *2017-2018* Semester : *Fall* Coordinator : *Fırat Arıkan* Date : *December 23, 2017* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :
	6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6		
S	O	L	U	T	I	ONS	SHOW YOUR WORK

(7+7+7 pts) **1.** Evaluate each of the following limits.

a) $\lim\limits_{x\to 1^+}\left(\dfrac{1}{x-1}-\dfrac{1}{e^x-e}\right)$ $(\infty-\infty)$

$$= \lim_{x\to 1^+} \frac{e^x-e-x+1}{(x-1)(e^x-e)} \quad \left(\frac{0}{0}\right)$$

$$\underset{L.R}{=} \lim_{x\to 1^+} \frac{e^x-1}{e^x-e+e^x(x-1)} = \infty \ (\text{dne})$$

Answer: $\boxed{\infty}$

b) $\lim\limits_{x\to 0^+}\left(\dfrac{2}{\pi}\arccos(x)\right)^{\frac{1}{x}}$ Let $y=\left(\frac{2}{\pi}\arccos(x)\right)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\left(\frac{2}{\pi}\arccos(x)\right)$

$$\Rightarrow \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{1}{x}\ln\left(\frac{2}{\pi}\arccos(x)\right) \quad \left(\frac{0}{0}\right)$$

$$\underset{L.R}{=} \lim_{x\to 0^+}\frac{-\frac{1}{\sqrt{1-x^2}}}{\arccos(x)} = \frac{-1}{\pi/2} = -\frac{2}{\pi}$$

$$\underset{\substack{\ln \text{ is} \\ \text{continuous}}}{\Longrightarrow} \ln\left(\lim_{x\to 0^+} y\right) = -\frac{2}{\pi} \Longrightarrow \lim_{x\to 0^+} y = e^{-2/\pi}$$

Answer: $\boxed{e^{-2/\pi}}$

c) $\lim\limits_{x\to 0^+}\dfrac{\int_0^{x^2}\sin(t^2)\,dt}{x^3}$ $\left(\frac{0}{0}\right)$

$$\underset{L.R}{=} \lim_{x\to 0^+}\frac{\sin x^4 \cdot 2\cancel{x}}{3x^{\cancel{2}}} = \frac{2}{3}\lim_{x\to 0^+}\frac{\sin x^4}{x} \quad \left(\frac{0}{0}\right)$$

(& FTC)

$$\underset{L.R}{=} \frac{2}{3}\lim_{x\to 0^+}\cos x^4 \cdot 4x^3 = 1\cdot 0 = 0$$

Answer: $\boxed{0}$

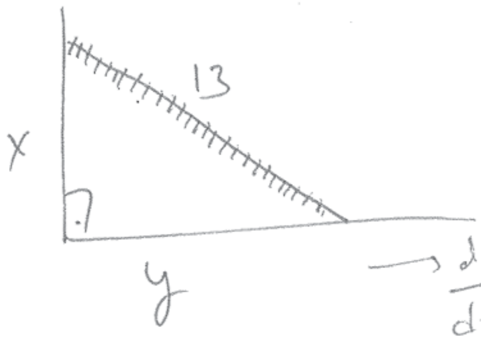
M E T U
Department of Mathematics

Group	Math 119 - Calculus with Analytic Geometry **Midterm 1**	List No.

Year : 2017-2018 Semester : Spring Instructors : A.B., E.C., M.A. Date : 31.03.2018 Time : 13:30 Duration : 110 minutes	Last Name : Name : Student No.: Department: Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6		

1. (25 pts.) A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/s. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?



$\left.\frac{dy}{dt}\right|_{t=t_0} = 0.6$

$$x^2 + y^2 = 169$$

$$\frac{d}{dt}: \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$$

$x = 12$ at $t = t_0$

$y = 5$ at $t = t_0$

$\left.\frac{dx}{dt}\right|_{t=t_0} = ?$

$$12\frac{dx}{dt} + 5\cdot(0.6) = 0$$

$$\frac{dx}{dt} = -\frac{3}{12} = -0.25 \ m/s$$

2. (15 pts.) Find the derivatives of the following functions.

(4 pts.) **a.** $f(x) = \sin(\sec(\tan x))$.

$$f'(x) = \cos(\sec(\tan x)) \cdot \sec(\tan x)\tan(\tan x) \cdot \sec^2 x$$

(7 pts.) **b.** $f(x) = (\sin x + \cos x)^{\sqrt{x}}$. $\ln f(x) = \sqrt{x} \cdot \ln(\sin x + \cos x)$

$$\frac{f'(x)}{f(x)} = \frac{\ln(\sin x + \cos x)}{2\sqrt{x}} + \sqrt{x} \cdot \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)$$

$$f'(x) = (\sin x + \cos x)^{\sqrt{x}} \left(\frac{\ln(\sin x + \cos x)}{2\sqrt{x}} + \sqrt{x} \cdot \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)\right)$$

(4 pts.) **c.** $f(x) = 1 + \arcsin(x^2) + 5^x - \log_5(x^3 - x)$.

$$f'(x) = \frac{2x}{\sqrt{1 - x^4}} + \ln 5 \cdot 5^x - \frac{1}{(x^3 - x) \cdot \ln 5} \cdot (3x^2 - 1)$$

3. (10 pts.) Find the equation of the normal line to the curve $y = x^2y^3 + x^3y^2$ at the point $(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}})$. By implicit differentiation,

$$y' = 2xy^3 + x^2 3y^2 y' + 3x^2y^2 + x^3 \cdot 2y \cdot y'$$

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$$

$$y'\left(\frac{1}{\sqrt[4]{2}}\right) = \frac{-5}{3}$$

Normal line equation:

$$y - \frac{1}{\sqrt[4]{2}} = \frac{3}{5}\left(x - \frac{1}{\sqrt[4]{2}}\right)$$

4. **(24 pts.)** Find the following limits.

(8) a. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-3}\right)^x = \lim\limits_{x\to\infty} e^{\cancel{x\ln}(1+\frac{6}{x-3})} = e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{6}{x-3}\right)}$ $(0\cdot\infty$ type$)$

$(1^\infty$ type$)$ as e^x is continuous

$= e^{\lim\limits_{x\to\infty}\frac{\ln(1+\frac{6}{x-3})}{1/x}} = e^{\lim\limits_{x\to\infty}\frac{\frac{-6}{(x-3)^2}/(1+\frac{6}{x-3})}{-1/x^2}} = e^6$

$\left(\frac{0}{0}\text{ type}\right)$ by l'Hospital's rule

(8) b. $\lim\limits_{x\to 0}\sqrt[x]{1-2x} = \lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\frac{1}{x}\ln(1-2x)} = e^{\lim\limits_{x\to 0}\frac{\ln(1-2x)}{x}}$

$(1^\infty$ type$)$ $\left(\frac{0}{0}\text{ type}\right)$

because e^x is continuous

$= e^{\lim\limits_{x\to 0}\frac{\frac{-2}{1-2x}}{1}} = e^{-2}$

by l'Hospital's rule

(8) c. $\lim\limits_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x} = \lim\limits_{x\to 0}\frac{5\cos 5x-3\cos 3x}{\cos x}$ by l'Hospital's rule

$\left(\frac{0}{0}\text{ type}\right)$

$= \frac{5\cos 0-3\cos 0}{\cos 0}$ because $\cos x$ is continuous at 0

$= 2.$

5. (16 pts.) Show that the equation $x^3 - 5x + 1 = 0$ has exactly three solutions. Let $f(x) = x^3 - 5x + 1$. Since it is polynomial, it is continuous on $\mathbb{R}$. So we can use IVT.

Since $f(-3) < 0$ and $f(-2) > 0$ & $f(0) > 0$ and $f(1) < 0$ & $f(2) < 0$ and $f(3) > 0$, by IVT there are roots $c_1 \in (-3,-2)$, $c_2 \in (0,1)$, $c_3 \in (2,3)$ s.t. $f(c_1) = f(c_2) = f(c_3) = 0$.

So there exist at least three solutions for $f(x)$.

$f'(x) = 3x^2 - 5$. So $f(x)$ is strictly increasing on $\left(-\infty, -\sqrt{\frac{5}{3}}\right)$ and on $\left(\sqrt{\frac{5}{3}}, \infty\right)$; and strictly decreasing on $\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

That's why $f(x)$ is one-to-one on $\left(-\infty, -\sqrt{\frac{5}{3}}\right)$, on $\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$ and on $\left(\sqrt{\frac{5}{3}}, \infty\right)$. Since $(-3,-2) \subseteq \left(-\infty, -\sqrt{\frac{5}{3}}\right)$, $(0,1) \subseteq \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$ and $(2,3) \subseteq \left(\sqrt{\frac{5}{3}}, \infty\right)$; c_1, c_2, c_3 are the unique solutions.

6. (10 pts.) For what values of the constants c and d is the function

$$f(x) = \begin{cases} cx + d & x \leq 3 \\ dx^2 - 4 & x > 3 \end{cases}$$

continuous and differentiable on $\mathbb{R} = (-\infty, \infty)$?

For $x \neq 3$ it is continuous & diff'ble since it is a polynomial.

For $x = 3$, $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = f(3)$; $f(3) = 3c + d$

$$\lim_{x \to 3^+} f(x) = \boxed{9d - 4 = 3c + d} = \lim_{x \to 3^-} f(x) = f(3)$$

For $x = 3$, $$f'(3) = \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3}$$

$$\lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^-} \frac{(cx + d) - (3c + d)}{x - 3} = c$$

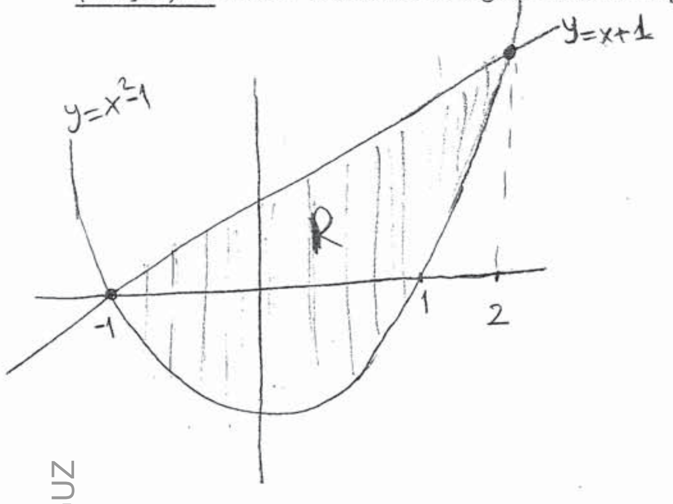
$$\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \frac{(dx^2 - 4) - (3c + d)}{x - 3} = \lim_{x \to 3^+} \frac{dx^2 - 4 - (9d - 4)}{x - 3} = 6d$$

$$\boxed{c = 6d} \ \& \ \boxed{9d - 4 = 3c + d} \Rightarrow \boxed{c = -2{,}4 \quad d = -0{,}4}$$

Math 119 Calculus With Analytic Geometry		
Midterm 2		
Acad. Year : 2017-2018 Semester : Spring Coordinator: Emre Coşkun Date : May 5th, 2018 Time : 13:30 Duration : 110 minutes	Last Name : Name : Student No. : Department :	Signature :
	6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	SHOW YOUR WORK	

(12 pts) **1.** Sketch the bounded region R between $y-x-1=0$ and $y-x^2+1=0$. Find the area of R.



Intersection points: $y=x+1$ and $y=x^2-1$

$x^2-1=x+1 \Rightarrow x=-1$ and $x=2$

Area of R: $\displaystyle\int_{-1}^{2}\left[(x+1)-(x^2-1)\right]dx =$

$= \displaystyle\int_{-1}^{2}(-x^2+x+2)\,dx = \left.-\frac{x^3}{3}+\frac{x^2}{2}+2x\right|_{-1}^{2}$

$=\left(-8/3+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)$

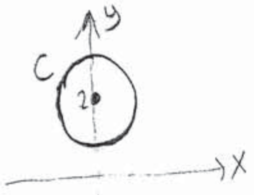
$=\frac{27}{6}$

(12 pts) **2.** Find the coordinates of the points on the curve $x^2+(y-2)^2=1$ that are farthest from the origin.

Distance function; $D(x,y)=\sqrt{x^2+y^2}$ Consider $D^2=x^2+y^2$

On the curve ; $D^2(y) = 1-(y-2)^2+y^2$

$= -3+4y$, $y\in[1,3]$



$2D.D' = 4 \Rightarrow D'\neq 0$ for any point on the curve. Then there is no critical point.

• $D'=\frac{2}{D}=\frac{2}{\sqrt{4y-3}} \Rightarrow D'$ is defined for all $y\in[1,3]$, so there is no singular point.

Consider end points, $[1,3]$

$D(1)=\sqrt{1-1+1}=1$

$D(3)=\sqrt{1-1+9}=3 \longrightarrow$ max. of D

• $y=3 \Rightarrow x=\sqrt{1-1^2}=0$

Thus, the farthest point $(0,3)$.

(27 pts) **3.** Evaluate the following integrals

(a) $\int \frac{x^3\,dx}{\sqrt[3]{x+1}}$ $\quad x+1=u^3 \quad x=u^3-1 \quad dx=3u^2\,du$

$x^3=(u^3-1)^3=u^9-3u^6+3u^3-1$

Then $\int \frac{(u^9-3u^6+3u^3-1)\,3u^2\,du}{u} = \int (3u^{10}-9u^7+9u^4-3u)\,du$

$\frac{3u^{11}}{11}-\frac{9u^8}{8}+\frac{9u^5}{5}-\frac{3u^2}{2}+C$

$\frac{3}{11}(x+1)^{11/3}-\frac{9}{8}(x+1)^{8/3}+\frac{9}{5}(x+1)^{5/3}-\frac{3}{2}(x+1)^{2/3}+C$

(b) $\int \frac{\sec x\,\tan^2 x}{\sin x}\,dx$ $= \int \frac{\sin^2 x}{\sin x\cdot\cos^3 x}\,dx = \int \frac{\sin x\,dx}{\cos^3 x}$

$\cos x=u \Rightarrow du=-\sin x\,dx$

So $\int \frac{-du}{u^3}=\int -u^{-3}\,du=\frac{u^{-2}}{2}+C=\frac{1}{2u^2}+C$

$=\frac{1}{2\cos^2 x}+C$

(c) $\int \frac{x^5+3x^4+5x^3+5x^2+3x+1}{x^2+x+1}\,dx$

$$
\begin{array}{l|l}
x^5+3x^4+5x^3+5x^2+3x+1 & x^2+x+1 \\
\hline
-\;x^5+x^4+x^3 & x^3+2x^2+2x+1 \\
\quad 2x^4+4x^3+5x^2+3x+1 & \\
\quad 2x^4+2x^3+2x^2 & \\
\quad\quad 2x^3+3x^2+3x+1 & \\
\quad -\;2x^3+2x^2+2x & \\
\quad\quad\quad x^2+x+1 & \\
\quad\quad -\;x^2+x+1 & \\
\quad\quad\quad 0 &
\end{array}
$$

$\therefore \int \frac{x^5+3x^4+5x^3+5x^2+3x+1}{x^2+x+1}\,dx=\int x^3+2x^2+2x+1$

$=\frac{x^4}{4}+\frac{2x^3}{3}+x^2+x+C$

(12 pts) 4. Evaluate $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{4(4j-2)}{n^2}\sqrt{(\frac{4j-2}{n})^2+1}$

Consider $f(x) = x\sqrt{x^2+1}$ and partition

$P_n = \{0, \frac{4}{n}, \frac{8}{n}, \cdots 4\}$ such that $\Delta x_j = \Delta x = \frac{4}{n}$, $j=1,2,\ldots n$

Take $c_j = \frac{4j-2}{n}$, $j=1,2,\ldots n$.

Then $\sum_{j=1}^{n} f(c_j)\Delta x_j = \sum_{j=1}^{n}\frac{4(4j-2)}{n^2}\sqrt{\left(\frac{4j-2}{n}\right)^2+1}$.

By the integral definition,

$$\lim_{n\to\infty}\sum_{j=1}^{n} f(c_j)\Delta x_j = \int_0^4 x\sqrt{x^2+1}\,dx = \int_1^{17}\frac{\sqrt{u}}{2}\,du = \frac{1}{3}\left(17^{3/2}-1\right)$$

(12 pts) 5. Decide whether $\int_e^\infty \frac{|\cos x|\,dx}{x(\ln x)^2}$ is convergent or not.

Since $\int_e^\infty \frac{|\cos x|\,dx}{x(\ln x)^2} < \int_e^\infty \frac{dx}{x(\ln x)^2} = \int_1^\infty \frac{du}{u^2} = 1 < \infty$,

by Comparison Theorem, the integral converges

KEY

(25 pts) **6.** Let $f(x) = \frac{1}{\ln x}$. **(a)** Find the domain of f.

$(0,\infty)\setminus\{1\} = (0,1)\cup(1,\infty)$

(b) Find the intervals of increase and decrease.

$$f'(x) = -\frac{1}{x\cdot(\ln x)^2} < 0 \quad \forall x \in \mathrm{Dom}(f)$$

f decreasing on: $(0,1)$, $(1,\infty)$

(c) Find the inflection point and the intervals where f is concave up and down.

$$f''(x) = \frac{2+\ln x}{x^2\cdot(\ln x)^3}$$

x	0		e^{-2}		1	
f''	‖	$+$	ϕ	$-$	‖	$+$

conc. up on: $(0,e^{-2})$, $(1,\infty)$

conc. down on: $(e^{-2},1)$

infl. pt: $x = e^{-2}$

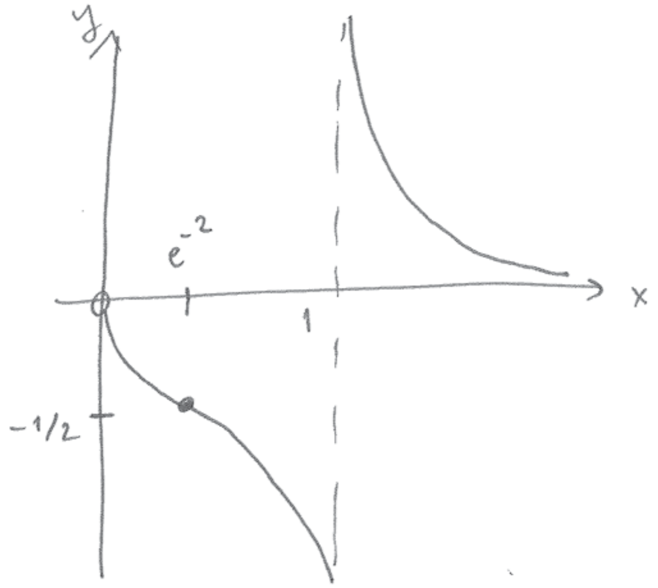
(d) Determine the vertical and horizontal asymptotes.

$\lim_{x\to 1^+} f(x) = \infty$ $\quad \lim_{x\to 1^-} f(x) = -\infty$: $x=1$ V.A.

$\lim_{x\to\infty} f(x) = 0$: $y=0$. H.A.

Note: $x=0$ is NOT H.A since $\lim_{x\to 0^+} f(x) = 0$

(e) Sketch the graph of f.

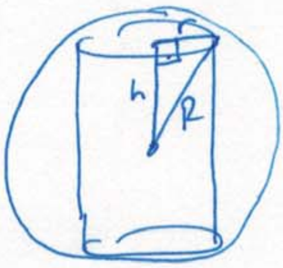


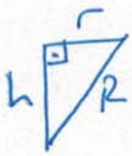
M E T U Department of Mathematics

Math 119 Calculus With Analytic Geometry Final	
Acad. Year : *2017-2018* Semester : *Spring* Coordinator: *Emre Coşkun* Date : *May 26th 2018* Time : *13:30* Duration : *120 minutes*	Last Name : Name : Signature : Student No. : Department :
	8 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

(10 pts) **1.** The cylinder with the maximum possible volume is inscribed in a ball with radius R. Find the dimensions of the cylinder.





$V = \pi r^2 h$

$V(h) = \pi \cdot (R^2 - h^2) h$

$= \pi h R^2 - \pi h^3$

$r^2 + h^2 = R^2$

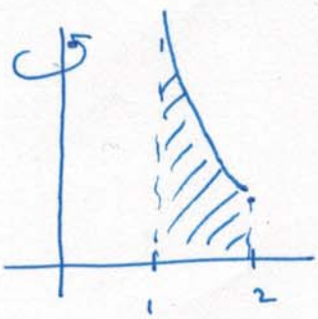
$r^2 = R^2 - h^2$

$0 = \frac{dV}{dh} = \pi R^2 - 3\pi h^2$

height: $2h$

$\Rightarrow h = \frac{R}{\sqrt{3}}$ $\quad r = \sqrt{\frac{2}{3}}\, R$

(15 pts) **2.** The curve $y = \frac{1}{\sqrt{x^2-1}}$, $1 < x \leq 2$, is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the solid of revolution that is generated.

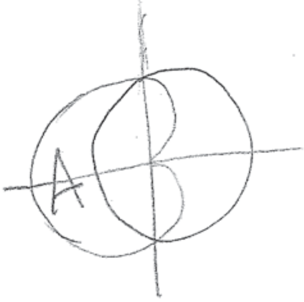


$V = \int_1^2 2\pi x \cdot f(x)\, dx = \int_1^2 2\pi x \cdot \frac{1}{\sqrt{x^2-1}}\, dx$

$= 2\pi \cdot \sqrt{x^2-1} \Big|_1^2 = 2\pi\sqrt{3}.$

(15 pts) **3.** The polar region R lies inside the cardioid $r = 1 - \cos\theta$ and outside the circle $r = 1$. Sketch and find the area of R.

Find the intersection points



$$1 - \cos\theta = 1 \Rightarrow \cos\theta = 0$$

$$\theta_1 = \frac{\pi}{2} \quad \text{and} \quad \theta_2 = \frac{3\pi}{2}$$

$$A = A_1 - A_2 \qquad A_1 = \frac{1}{2}\int_{\pi/2}^{3\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$A_2 = \frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2} \qquad = \frac{1}{2}\left(\frac{\theta}{2} - \sin\theta + \frac{\theta + \cos 2\theta}{2}\right)\Big|_{\pi/2}^{3\pi/2}$$

$$= \frac{3\pi}{4} + 2$$

$$A = \left(\frac{3\pi}{4} + 2\right) - \frac{\pi}{2} = \frac{\pi}{4} + 2$$

(10 pts) **4.** Find the length of the curve given by the polar equation: $r = ae^{m\theta}$ (where $a > 0$ and $m > 0$ are constants) for $-\infty < \theta \le 0$.

$$L = \int_{-\infty}^{0} \sqrt{(ae^{m\theta})^2 + (mae^{m\theta})^2} \, d\theta$$

$$= \int_{-\infty}^{0} ae^{m\theta}\sqrt{1+m^2} \, d\theta$$

$$= a\sqrt{1+m^2}\,\frac{e^{m\theta}}{m}\Big|_{-\infty}^{0} = \frac{a\sqrt{1+m^2}}{m}$$

(7+ 9+ 9=25 pts) **5.** Evaluate the following integrals.

(a) $\int_1^{e^2} x \ln x dx = I$

Let $\ln x = u$ & $x\,dx = dv$. Then $\frac{dx}{x} = du$, $\frac{x^2}{2} = v$

So

$$I = \left[\frac{x^2 \ln x}{2} - \int \frac{x\,dx}{2}\right]_{x=1}^{x=e^2} = \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right]_{x=1}^{x=e^2} = \frac{3e^4+1}{4}$$

by integration by parts.

(b) $\int \cos(\sqrt{x})dx = I$

Let $\sqrt{x} = t$. So $x = t^2$ & $dx = 2t\,dt$. So $I = \int 2\cos(t).t\,dt$

Let $t = u$ & $\cos(t)dt = dv$. Then $dt = du$ & $\sin t = v$.

Applying integration by parts,

$$I = 2\cdot\left[t\cdot \sin t - \int \sin t\,dt\right] = 2[t\sin(t) + \cos(t)) + C \Longrightarrow$$

$$\Longrightarrow I = 2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$

(c) $\int \frac{2x^2 - x + 4}{x^3 - x^2 + 4x - 4}dx = I$ By the method of partial fraction,

$$I = \int \frac{2x^2 - x + 4}{(x^2+4)(x-1)}dx \Rightarrow \frac{2x^2 - x + 4}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C_1}{x-1}$$

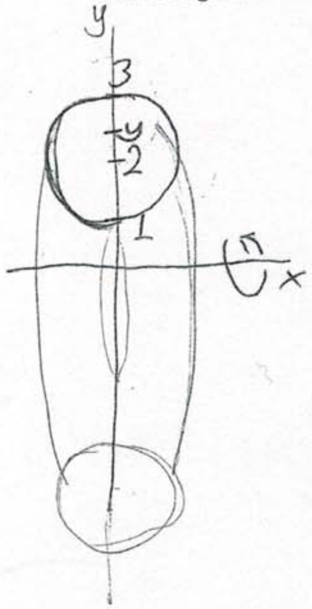
After correct calculations, $A = 1$, $B = 0$, $C_1 = 1$

So

$$I = \int \left(\frac{x+0}{x^2+4} + \frac{1}{x-1}\right)dx = \int \frac{x\,dx}{x^2+4} + \int \frac{dx}{x-1}$$

$$I = \frac{1}{2}\ln|x^2+4| + \ln|x-1| + C$$

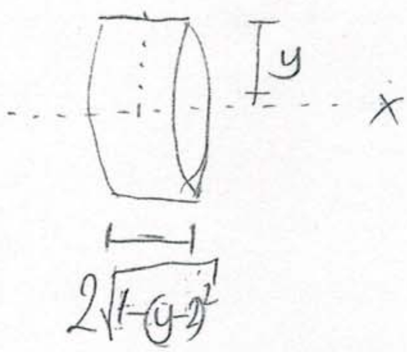
(10 pts) **6.** The disc $x^2 + (y-2)^2 = 1$ is rotated about ~~x~~ y-axis. Find the volume of solid of revolution that is generated.



Cylindrical Shell Method:

$x^2 + (y-2)^2 = 1$

$\Rightarrow x = \mp\sqrt{1-(y-2)^2}$



$$V = 2\int_1^3 2\pi y\sqrt{1-(y-2)^2}\,dy = 2\int_1^3 2\pi(y-2+2)\sqrt{1-(y-2)^2}\,dy$$

$$= \underbrace{4\pi\int_1^3 (y-2)\sqrt{1-(y-2)^2}\,dy}_{\substack{u=1-(y-2)^2,\ du=-2(y-2)dy \\ u(1)=0 \quad u(3)=0 \\ =0}} + \underbrace{8\pi\int_1^3 \sqrt{1-(y-2)^2}\,dy}_{\substack{y-2=\sin\theta \\ dy=\cos\theta\, d\theta \\ \sqrt{1-(y-2)^2}=\cos\theta \\ \theta=\arcsin(y-2) \\ \theta(1)=-\frac{\pi}{2} \quad \theta(3)=\frac{\pi}{2}}} = 8\pi\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$$

$$= 16\pi\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= 8\pi\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = 4\pi^2$$

(8 pts) **7.** Evaluate $\lim\limits_{x\to\infty} \dfrac{\int_0^x (\arctan t)^2\,dt}{\sqrt{x^2+1}}$.

$$\lim_{x\to\infty} \frac{\int_0^x (\arctan t)^2\,dt}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\int_0^x (\arctan t)^2\,dt}{\frac{d}{dx}\left(\sqrt{x^2+1}\right)}$$ by l'Hospital's rule since both limits exist.

by F.T.C

$$= \lim_{x\to\infty} \frac{(\arctan x)^2}{\frac{x}{\sqrt{x^2+1}}} = \frac{(\pi/2)^2}{1} = \frac{\pi^2}{4}.$$

(7 pts) **8.** Approximate $\arctan(1.02)$.

$f(x) = \arctan x \quad a = 1 \quad x = 1.02 \quad f'(x) = \frac{1}{1+x^2}$

$f(x) \approx f(a) + f'(a)(x-a)$ (tangent line approximation)

$f(a) = \arctan 1 = \frac{\pi}{4} \quad f'(a) = \frac{1}{2} \quad x-a = 0.02 = \frac{1}{50}$

$f(1.02) \approx \frac{\pi}{4} + \frac{1}{100}.$

SOLUTIONS

(8+8+8 pts) **2.** Evaluate each of the following integrals.

a) $I = \int \frac{3x^2 - 2x + 4}{(x-1)(x^2+4)} dx$ $\qquad \frac{3x^2-2x+4}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{(x^2+4)A + (x-1)(Bx+C)}{(x-1)(x^2+4)}$

$\Rightarrow 3x^2 - 2x + 4 = (A+B)x^2 + (C-B)x + 4A - C$

$\Rightarrow \begin{cases} (1)\ A+B=3 \\ (2)\ C-B=-2 \\ (3)\ 4A-C=4 \end{cases}$ $\left.\begin{matrix} A+C=1 \\ (4) \end{matrix}\right\}$ $5A=5 \Rightarrow \boxed{A=1} \xRightarrow{(4)} \boxed{C=0} \xRightarrow{(2)} \boxed{B=2}$

So, $I = \int \left(\frac{1}{x-1} + \frac{2x}{x^2+4}\right) dx = \ln|x-1| + \ln|x^2+4| + C$

b) $I = \int_0^{\ln 2} \frac{e^{2x}}{e^{4x} + 2e^{2x} + 1} dx$ $\qquad$ Let $u = e^{2x} \Rightarrow du = 2e^{2x}dx$

Also $x=0 \Rightarrow u=1$ & $x=\ln 2 \Rightarrow u=4$

$\Rightarrow I = \frac{1}{2} \int_1^4 \frac{du}{(u+1)^2} = -\frac{1}{2} \frac{1}{u+1} \Big|_1^4$

$= -\frac{1}{2}\left[\frac{1}{5} - \frac{1}{2}\right] = -\frac{1}{10} + \frac{1}{4} = \frac{3}{20}$

Answer: $\boxed{3/20}$

c) $I = \int_{-1}^{1} (\sin(x)\ln(1+x^2) + e^x)\, dx$ $\qquad \int_{-1}^{1} \underbrace{\sin(x)\ln(1+x^2)}_{f(x)} dx = 0$ since $f(x)$ is odd (finite)

Also $\int_{-1}^{1} e^x dx = e^x \Big|_{-1}^{1} = e - e^{-1}$ (finite). So, both integral exist.

Answer: $\boxed{e - e^{-1}}$

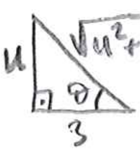
$\Rightarrow I = \underbrace{\int_{-1}^{1} \sin(x)\ln(1+x^2)dx}_{0} + \int_{-1}^{1} e^x dx = e - e^{-1}$

(8+8 pts) **3.** Evaluate each of the following integrals.

a) $I = \int \frac{x}{\sqrt{x^4 + 4x^2 + 13}} dx = \int \frac{x\, dx}{\sqrt{(x^2+2)^2 + 3^2}}$. So, let $u = x^2+2 \Rightarrow du = 2x\,dx$

$\Rightarrow I = \frac{1}{2} \int \frac{du}{\sqrt{u^2+3^2}}$. Now let $u = 3\tan\theta \Rightarrow du = 3\sec^2\theta\, d\theta$

$\Rightarrow I = \frac{1}{2} \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$



$\Rightarrow I = \frac{1}{2}\ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{(x^2+2)^2+9}}{3} + \frac{x^2+2}{3}\right| + C$

b) For $x > 0$, $I = \displaystyle\int x \arcsin\left(\frac{1}{x}\right) dx$

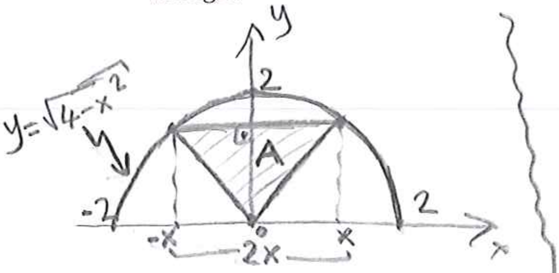
Let $u = \arcsin\left(\frac{1}{x}\right)$ $\Rightarrow \begin{cases} du = \dfrac{1}{\sqrt{1-\frac{1}{x^2}}}\cdot\left(-\dfrac{1}{x^2}\right)dx \\ v = \dfrac{x^2}{2} \end{cases}$

$\dfrac{-|x|}{x^2\sqrt{x^2-1}} = \dfrac{-x}{x^2\sqrt{x^2-1}}$ (as $x>0$)

Integration by parts $\Longrightarrow$ $I = \arcsin\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right) + \displaystyle\int \frac{x^2}{2}\cdot\frac{x}{x^2\sqrt{x^2-1}}\,dx$

$= \frac{1}{2}\arcsin\left(\frac{1}{x}\right)x^2 + \frac{1}{2}\left(x^2-1\right)^{1/2} + C$

(10 pts) 4. Consider a triangle in the xy-plane with two vertices on the curve $y = \sqrt{4-x^2}$ and one vertex at the origin, which has an edge parallel to the x-axis. What is the largest possible area of this triangle?



Area $= A = \dfrac{2x\cdot\sqrt{4-x^2}}{2} = x\cdot\sqrt{4-x^2}$

Consider $A(x) = x\sqrt{4-x^2}$ on $[0,2]$

To maximize $A(x)$, we maximize

$f(x) = (A(x))^2 = x^2(4-x^2)$ on $[0,2]$:

$f'(x) = 4x(2-x^2) = 0 \Rightarrow \underbrace{\boxed{x=0,\ x=\sqrt{2}}}_{\text{crit pts}},\ \cancel{x=-\sqrt{2}}$

$\left.\begin{array}{l} f(0)=0 \\ f(\sqrt{2})=4 \\ f(2)=0 \end{array}\right\} \Rightarrow 4 = $ Abs max of $f(x)$ on $[0,2]$

$\therefore \sqrt{4} = 2 =$ Abs max of $A(x)$ on $[0,2]$

Or Consider $f(x)$ on $(0,2)$

Then still $f(\sqrt{2}) = 4$

Also $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 2^-} f(x) = 0$

$\Rightarrow 4 =$ Abs max of $f(x)$ on $(0,2)$.

$\therefore \sqrt{4} = 2 =$ Abs max of $A(x)$ on $(0,2)$

Answer: 2

(10 pts) 5. Using the definition of definite integral in terms of Riemann sums, show that for any continuous **increasing** function $f(x)$ defined on $\mathbb{R}$ such that $f(0) = 0$, $f(1/4) = 2$, $f(1/2) = 4$, $f(3/4) = 50$ and $f(1) = 120$, we have that

$$14 \leq \int_0^1 f(x)dx \leq 44$$

0, 1/4, 1/2, 3/4, 1

$n=4$

We use right endpoints to compute the upper sum of f: $\boxed{\text{as } f \text{ is } \nearrow}$

$\mathcal{U}(f,P_4) = \left[f(1/4) + f(1/2) + f(3/4) + f(1)\right]\cdot\dfrac{1-0}{4}$

$= \left[2+4+50+120\right]\frac{1}{4} = 44.$

We use left endpoints to compute the lower sum of f:

$\mathcal{L}(f,P_4) = \left[f(0) + f(\frac{1}{4}) + f(1/2) + f(\frac{3}{4})\right]\cdot\dfrac{1-0}{4}$

$= \left[0+2+4+50\right]\cdot\frac{1}{4} = 14.$

Since f is cont, its integral exists and satisfies $\mathcal{L}(f,P_4) \leq \int_0^1 f(x)dx \leq \mathcal{U}(f,P_4)$

$\therefore 14 \leq \int_0^1 f(x)dx \leq 44$

$\square$.

SOLUTIONS

(5+5+5+4 pts) **6.** Let $f(x) = \dfrac{(3x+1)^2}{(x-1)^2}$. You are given that

$$f'(x) = \frac{-8(3x+1)}{(x-1)^3} \qquad \text{and} \qquad f''(x) = \frac{48(x+1)}{(x-1)^4}$$

a) Find all local maxima and local minima of $f(x)$; and determine intervals of increase/decrease.

EXPLANATIONS: $f'(x) = 0 \Rightarrow \boxed{x = -\tfrac{1}{3} \text{ critical pt.}}$

x		$-1/3$		1	
f'	$-$	0	$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

$\boxed{\text{loc. min at } -1/3 \ (\text{by } 1^{st}\text{DT})}$ $\boxed{\text{no local max.}}$

$\boxed{f'(1) \text{ dne but } x=1 \text{ is not a singular pt since } 1 \notin Dom(f) \Rightarrow \text{no singular pt}}$

local max (if any) $\boxed{\text{none}}$

local min (if any) $\boxed{\text{at } -1/3}$

increasing on $\boxed{(-1/3, 1)}$

decreasing on $\boxed{(-\infty, -1/3)}$, $(1, \infty)$

b) Determine all inflection point(s) and the intervals where $f(x)$ concave up and concave down.

EXPLANATIONS: $f''(x) = 0 \Rightarrow \boxed{x=-1}$ Candidate for an inflection pt

x		-1		1	
f''	$-$	0	$+$		$+$
f	conc. down		conc. up		conc. up

$\boxed{f'(-1) \text{ finite} \Rightarrow \text{tangent line exists at } -1. \text{ Also one needs to check concavities}}$

$\Rightarrow \boxed{(-1, f(-1)) \text{ is the only inflection point}}$

inflection point(s) $\boxed{(-1, f(-1))}$

concave up on $\boxed{(-1, 1)}$, $(1, \infty)$

concave down on $\boxed{(-\infty, -1)}$

c) Determine all asymptotes of $f(x)$.

EXPLANATIONS:

$\lim_{x \to 1^+} f(x) = \infty = \lim_{x \to 1^-} f(x) \Rightarrow \boxed{x=1 \text{ vertical asymp.}}$

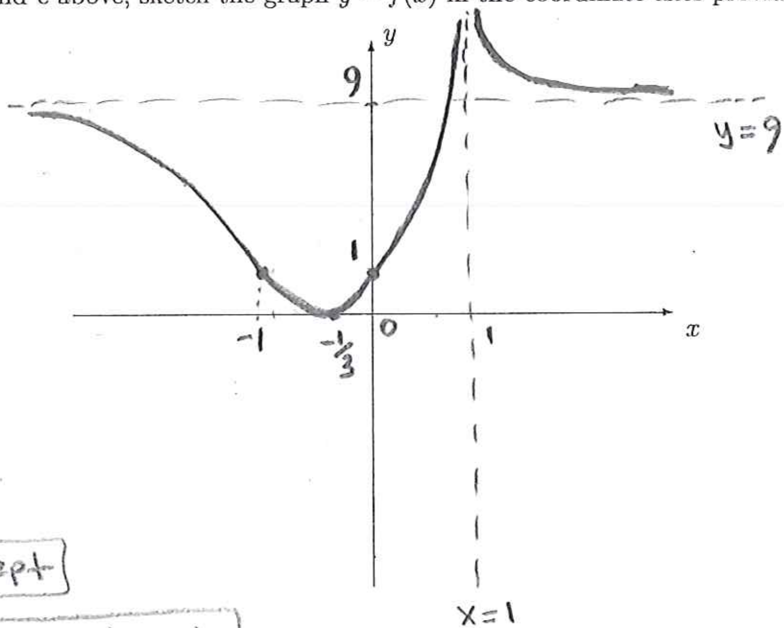
$\lim_{x \to \pm\infty} f(x) = 9 \Rightarrow \boxed{y = 9 \text{ horizontal asymp.}}$

horizontal (if any) $\boxed{y=9}$

vertical (if any) $\boxed{x=1}$

oblique (if any) $\boxed{\text{none}}$

d) Using the parts a, b and c above, sketch the graph $y = f(x)$ in the coordinate axes provided below.



$x=0 \Rightarrow y = f(0) = 1$

$\Rightarrow \boxed{y=1 \text{ y-intercept}}$

$y = f(x) = 0 \Rightarrow \boxed{x = -\tfrac{1}{3} \text{ x-intercept}}$

M E T U
Department of Mathematics

	Calculus with Analytic Geometry **Final Exam**	
Code : MATH 119 Acad.Year : 2017 - 2018 Semester : Fall Coord. : Fırat Arıkan	Last Name : Name : Dept. : Signature :	Stud. No : Sec. No :
Date : 11.01.2018 Time : 09.30 Duration : 135 minutes	8 Questions on 4 Pages Total **100** Points	

1	2	3	4	5	6	7	8		
S	O	L	U	T	I	O	N	S	

Q.1 (5+10 = 15 pts) Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} (x^2 - x - 1)\,e^x & \text{if } -3 \le x \le 0 \\ 2x - 1 & \text{if } 0 < x \le 1 \end{cases}$$

(a) Explain why $f(x)$ has both absolute maximum and absolute minimum values.

On both side of $x=0$, $f(x)$ is a product of continuous functions. Also, since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = -1 = f(0)$, $f(x)$ is continuous at $x=0$. Hence, $f(x)$ is continuous on the closed and bounded interval $[-3,1]$ so, by "max-min thm" f takes on absolute extreme values on $[-3,1]$.

(b) Find the absolute maximum and minimum values of $f(x)$.

f takes on exteme values at CPs or SPs or boundary points (BPs). From the derivative fnc.

$$f'(x) = \begin{cases} (x^2+x-2)e^x, & x \in (-3,0) \\ 2, & x \in (0,1) \end{cases}$$

we see that

CPs : $f'(x) = 0$ at $\boxed{x=-2}$ and $x=1$ (outside of $x \in (-3,0)$)

SPs : $f'(x)$ d.n.e at $\boxed{x=0}$ since $\lim_{x\to 0^-} \frac{(x^2-x-1)e^x-(-1)}{x} \overset{!!}{=} -2 \neq 2 = \lim_{x\to 0^+} \frac{2x-1-(-1)}{x}$

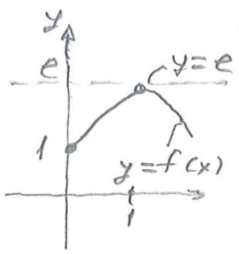
(The limit on the left follows e.g. from L'Hospital)

Comparing the fnc. values $f(-2) = 5/e^2$, $f(0) = -1$, $f(-3) = 11/e^3$ and $f(1) = 1$, we see that

$$f_{max} = f(1) = 1 \quad \text{and} \quad f_{min} = f(0) = -1$$

are the abs. max. and abs. min. values of f, respectively.

Q.2 (10 pts) For what values of the numbers a and b is the curve $f(x) = e^{ax^2+bx}$ **tangent** to the line $y = e$ at the point $(x,y) = (1,e)$?



The point $(x,y) = (1,e)$ must be on the curve, i.e. $f(1) = e^{a+b} = e$ giving ① $a+b=1$. The derivative must be ZERO at $x=1$, i.e. $f'(1) = (2ax+b)e^{ax^2+bx}\Big|_{x=1} = (2a+b)e^{a+b} = 0$

implying ② $2a+b=0$

Solving ① and ②, we find that $\boxed{a=-1}$ and $\boxed{b=2}$.

SOLUTIONS

Q.3 (2×7= 14 pts) Evaluate the following integrals.

(a) $I = \int_0^{\pi/2} (1+\sin^4 x)\cos^3 x \, dx = \int_0^{\pi/2} (1+\sin^4 x)(1-\sin^2 x)\cos x \, dx$

Put $\begin{cases} t = \sin x \\ dt = \cos x \, dx \\ t: 0 \to 1 \end{cases}$, we have $I = \int_0^1 (1+t^4)(1-t^2)\,dt = \int_0^1 (1-t^2+t^4-t^6)\,dt$

and

$$I = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 \Big|_0^1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

$$\boxed{I = \frac{76}{105}}$$

(b) $I = \int_1^4 e^{\sqrt{x}} \, dx$ Put $t = \sqrt{x}$, $dt = \frac{1}{2\sqrt{x}} dx = \frac{1}{2t} dx$ or $dx = 2t\,dt$ $(t: 1 \to 2)$

So $I = 2\int_1^2 t e^t \, dt = 2te^t \Big|_1^2 - 2\int_1^2 e^t \, dt = 4e^2 - 2e - 2e^t \Big|_1^2$

$\left\{ \begin{array}{ll} \text{On integrating by parts} \\ u = 2t & dv = e^t dt \\ du = 2dt & v = e^t \end{array} \right\}$

$I = 4e^2 - 2e - 2(e^2 - e)$

$$\boxed{I = 2e^2}$$

Q.4 (3×7= 21 pts) *This problem has 3 unrelated parts about* **improper** *integrals:*

(a) Evaluate the improper integral $I = \int_0^\infty x e^{-x^2} \, dx$. $I = \lim_{R\to\infty} I_R$, where

$$I_R = \int_0^R x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_0^R = -\frac{1}{2}(e^{-R^2} - 1) = \frac{1}{2}(1 - e^{-R^2})$$

Hence $I = \lim_{R\to\infty} \int_0^R x e^{-x^2} dx = \frac{1}{2} \lim_{R\to\infty} (1 - e^{-R^2}) = \frac{1}{2}$

$$\boxed{I = \frac{1}{2}}$$

(b) Determine whether the integral $\int_0^\infty \frac{1}{(x+e^x)^{1/3}} \, dx$ is convergent or divergent.

For $x \geq 0$, $x + e^x \geq e^x$ impliying that $\frac{1}{x+e^x} \leq \frac{1}{e^x}$ and that

$$0 \leq \frac{1}{(x+e^x)^{1/3}} \leq \frac{1}{e^{\frac{1}{3}x}}.$$

Also we have $\int_0^\infty e^{-x/3} dx = \lim_{R\to\infty} \int_0^R e^{-x/3} dx = \lim_{R\to\infty} -3(e^{-R/3} - e^0) = 3.$

So, $\int_0^\infty \frac{1}{(x+e^x)^{1/3}} dx$ is convergent by comparison test.

SOLUTIONS

(c) Find the values of p for which the integral $\displaystyle\int_0^1 \frac{dx}{(\tan x)^p}$ is convergent.

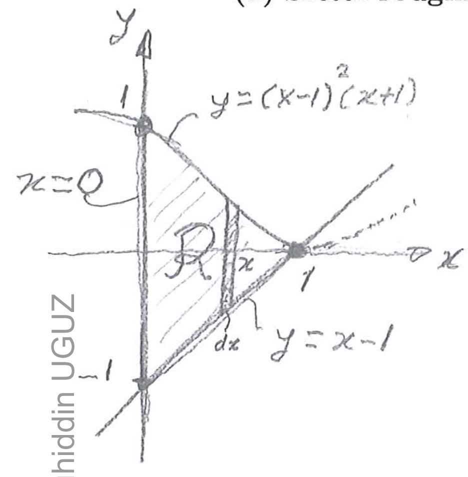
$$\lim_{x\to 0^+} \frac{\frac{1}{(\tan x)^p} > 0}{1/x^p > 0} = \lim_{x\to 0^+} \frac{x^p}{(\sin x)^p}(\cos x)^p = 1 \text{ for all } p.$$

Recall $\displaystyle\int_0^1 \frac{dx}{x^p}$ is convergent $\Leftrightarrow p<1$

So, by limit comparison test, $\displaystyle\int_0^1 \frac{dx}{(\tan x)^p}$ is convergent $\Leftrightarrow p<1$.

Q.5 (7+4+4= 15 pts) Let $\mathcal{R}$ be the region bounded by the lines $x=0$ and $y=x-1$, and the curve $y=(x-1)^2(x+1)$ for $x\geq 0$. (*Do NOT evaluate integrals.*)

(a) Sketch **roughly** the region $\mathcal{R}$, and then express its **area** as a definite integral.



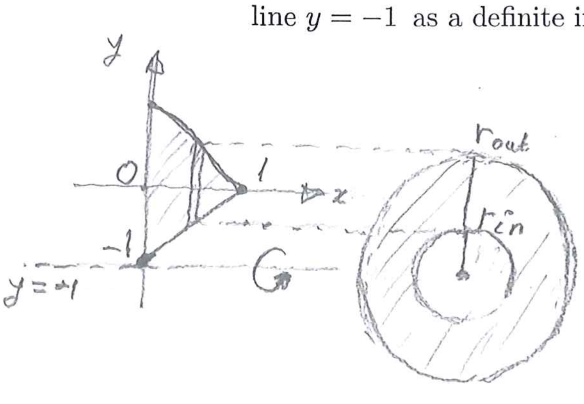
$$\text{Area of } R = \int_0^1 \left[(x-1)^2(x+1)-(x-1)\right]dx$$

(b) Express the **volume** of the solid obtained by rotating the region $\mathcal{R}$ about the y–axis as a definite integral.

By "cylindrical shell method",

$$V = 2\pi\int_0^1 x\left[(x-1)^2(x+1)-(x-1)\right]dx$$

(c) Express the **volume** of the solid obtained by rotating the region $\mathcal{R}$ about the line $y=-1$ as a definite integral.



$$dV = \pi\left[r_{out}^2 - r_{in}^2\right]dx$$

$r_{out} = (x-1)^2(x+1)-(-1)$, $r_{in} = x-1-(-1) = x$

By disk (cross-section) method,

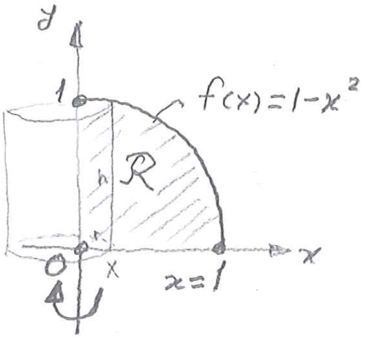
$$V = \pi\int_0^1 \left\{\left[(x-1)^2(x+1)+1\right]^2 - x^2\right\}dx$$

SOLUTIONS

Q.6 (7 pts) A plane region $\mathcal{R}$ is rotated about a line ℓ to obtain a solid $\mathcal{S}$. Then the **volume** V of solid $\mathcal{S}$, calculated by cylindrical shell method, is found to be

$$V = 2\pi \int_0^1 (x - x^3)\, dx.$$

Describe region $\mathcal{R}$ and find line ℓ.



Cylindrical shell method consists of

(a) $V = 2\pi \int_a^b r(x)\, h(x)\, dx$ (b) Shell will be parallel to the rotation axis.

(for all $x \in [a,b]$ there is a shell that is vertical)

so, we may take

$$V = 2\pi \int_0^1 x(1-x^2)\, dx$$

which means $h(x) = 1-x^2$, $a=0$, $b=1$ and the rotation axis is the line $x=0$. (i.e., y-axis)

Q.7 (8 pts) Find the **arclength** of the curve given by $y = f(x) = \int_1^x \sqrt{t^3 - 1}\, dt$ from $x = 1$ to $x = 2$.

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx$$

By FTC, $f'(x) = \sqrt{x^3 - 1}$ and $L = \int_1^2 \sqrt{1 + x^3 - 1}\, dx = \int_1^2 x^{3/2}\, dx$.

Thus

$$L = \frac{2}{5} x^{5/2} \Big|_1^2 = \frac{2}{5}\left(2^{5/2} - 1^{5/2}\right) = \frac{2}{5}\left(4\sqrt{2} - 1\right).$$

Q.8 (10 pts) Consider the function $y = y(x)$ defined implicitly by the equation $\tan(x+y) = xy$, where $y(\pi) = 0$. Use a linear approximation to estimate the value of this function at $x = \pi + 0.05$, i.e. calculate $y(\pi + 0.05)$ approximately.

A suitable linear approximation is of the form

$$L(x) = y(\pi) + y'(\pi)(x - \pi) = 0 + y'(\pi)(x - \pi)$$

By implicit differentiation,

$$\left.\frac{1 + y'(x)}{\cos^2(x + y(x))}\right|_{x=\pi} = y(x) + x y'(x)\Big|_{x=\pi} \Rightarrow \frac{1 + y'(\pi)}{\cos^2(\pi + y(\pi))} = y(\pi) + \pi y'(\pi)$$

or

$$\frac{1 + y'(\pi)}{\cos^2 \pi} = \pi y'(\pi) \Rightarrow 1 + y'(\pi) = \pi y'(\pi) \Rightarrow y'(\pi) = \frac{1}{\pi - 1}$$

Therefore, $L(x) = \dfrac{x - \pi}{\pi - 1}$ and $y(\pi + 0.05) \cong L(\pi + 0.05)$

$$y(\pi + 0.05) \cong \frac{\pi + 0.05 - \pi}{\pi - 1} = \frac{0.05}{\pi - 1} = \frac{1}{20(\pi - 1)}.$$

M E T U
Department of Mathematics

<table>
<tr><td></td><td colspan="2">Calculus with Analytic Geometry
First Midterm Exam</td><td></td></tr>
<tr><td>Code : MATH 119
Acad.Year : 2016
Semester : Fall
Coord. : Muhiddin Uğuz

Date : 19.11.2016
Time : 9.30
Duration : 110 minutes</td><td colspan="3">Last Name :
Name : Stud. No :
Dept. : Sec. No :
Signature :

6 Questions on 4 Pages
Total 100 Points</td></tr>
<tr><td>Q1 | Q2 | Q3 | Q4 | Q5 | Q6</td><td colspan="2">SHOW YOUR WORK !</td><td></td></tr>
</table>

Q.1 (15 pts) Show that the equation

$$\arctan x = 2 - x - x^3$$

has exactly ONE solution.

Consider the function $f(x) = \arctan x - 2 + x + x^3$

Note that $f(0) = -2 < 0$

$f(1) = \frac{\pi}{4} > 0$

Moreover f is continuous on $[0,1]$, and hence by I.V.T. (Intermediate Value Theorem), there exists $c \in (0,1)$ such that $f(c) = 0$. Hence given equation has at least one solution.

For the uniqueness of the solution;

Note that $f'(x) = \frac{1}{1+x^2} + 1 + 3x^2 \cancel{\geq} > 0 \quad \forall x$.

so $f(x)$ is a strictly increasing function and hence one-to-one.

Therefore the equation $f(x) = 0$ has at most one solution

Hence $f(x) = 0$ has exactly one solution.

OR/ since f is continuous on $[0,1]$ and differentiable on $(0,1)$, we can use Rolle's Thm:

if $f(x_0) = 0 = f(x_1)$ for some $x_0, x_1 \in (0,1)$ then $\exists\, r$ between x_0 & x_1 s.t. $f'(r) = 0$. But $f'(r) = \frac{1}{1+r^2} + 1 + 3r^2$ and hence $f'(r) \neq 0 \ \forall r$.

Q.2 (4 × 6 = 24 pts) Without using L'Hospital's rule, evaluate the limits:

(a) $\lim\limits_{x\to 0} \dfrac{\tan 5x}{\sin 3x}$ $= \lim\limits_{x\to 0} \left[\dfrac{\sin 5x}{5x} \cdot \dfrac{3x}{\sin 3x} \cdot \dfrac{1}{\cos 5x} \cdot \dfrac{5}{3} \right]$

since each limit exists $\searrow$ $= \lim\limits_{u\to 0} \dfrac{\sin u}{u} \cdot \lim\limits_{w\to 0} \dfrac{w}{\sin w} \cdot \lim\limits_{x\to 0} \dfrac{1}{\cos 5x} \cdot \dfrac{5}{3}$

$= 1 \cdot 1 \cdot 1 \cdot \dfrac{5}{3} = \dfrac{5}{3}$

(b) $\lim\limits_{x\to -\infty} (x - \sqrt{x^2 + 4x})$

Note that $x - \sqrt{x^2+4x} \le x \quad \forall\, x \le -4$ and $\lim\limits_{x\to -\infty} x = -\infty$

so $\lim\limits_{x\to -\infty} x - \sqrt{x^2+4x} = -\infty$

(c) $\lim\limits_{x\to 0} \dfrac{\cos[(x+2)^2] - \cos 4}{x}$ $= f'(0)$ where $f(x) = [\cos(x+2)^2]$ and hence

(or, you may choose $g(x) = \cos(x^2)$ and find $g'(2)$)

$f'(x) = -2(x+2)\sin[(x+2)^2]$

$= -2(0+2)\sin((0+2)^2) = -4\sin(4)$

(d) $\lim\limits_{x\to 0} x^2\, e^{\sin(\frac{\pi}{x})}$

e^x is increasing

$-1 \le \sin(\frac{\pi}{x}) \le 1 \implies e^{-1} \le e^{\sin(\frac{\pi}{x})} \le e$

$\implies x^2 e^{-1} \le x^2 e^{\sin(\frac{\pi}{x})} \le x^2 e$

since $\lim\limits_{x\to 0} \dfrac{x^2}{e} = 0 = \lim\limits_{x\to 0} x^2 e$, by Squeeze Thm, $\lim\limits_{x\to 0} x^2 e^{\sin(\frac{\pi}{x})} = 0$

Q.3 (10 pts) Let $f(x) = \begin{cases} x^2 + \arcsin x & \text{for } x \ge 0 \\ x + e^{x^2} & \text{for } x < 0 \end{cases}$. Find $f'(0)$ if it exists, or explain why it does NOT exist.

$f(0) = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x + e^{x^2}) = 1 \ne 0 \implies f$ is not continuous and hence not differentiable at $x = 0$.

(OR $\lim\limits_{t\to 0^+} \dfrac{f(t)-f(0)}{t} = \lim\limits_{t\to 0^+} \dfrac{t^2 + \arcsin t}{t} = \dots = 1$

$\lim\limits_{t\to 0^-} \dfrac{f(t)-f(0)}{t} = \lim\limits_{t\to 0^-} \dfrac{t+e^{t^2}}{t} = \dots = -\infty$

$\therefore f$ is not diff'ble at $x = 0$)

Q.4 (5 × 6 = 30 pts)

(a) By definition, the limit $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$ represents the derivative of a function $f(x)$ at the number $x = x_0$. Find $f(x)$ and x_0.

$f(x) = \sqrt[4]{x}$, $x_0 = 16$

[or $g(x) = \sqrt[4]{16+x}$ and $x_0 = 0$
or
⋮]

(b) Find $f'(x) = \dfrac{dy}{dx}$ if $y = f(x) = \dfrac{x^{\pi} + \cos x}{1 + x^3}$.

$$f'(x) = \frac{(\pi x^{\pi-1} - \sin x)(1+x^3) - (x^{\pi} + \cos x)(3x^2)}{(1+x^3)^2}$$

(c) Find $f'(x) = \dfrac{dy}{dx}$ if $y = f(x) = \sec(\sec x)$.

Recall that $(\sec x)' = \sec x . \tan x$

$f'(x) = \sec(\sec x) . \tan(\sec x) . \sec x . \tan x$

(d) Find $f'(x) = \dfrac{dy}{dx}$ if $y = f(x) = 3^x + \log_x 3$.

$y = 3^x \to \ln y = x \ln 3$
$\to \dfrac{y'}{y} = \ln 3$
$\to y' = y \ln 3 = 3^x \ln 3$

$y = \log_x 3 = \dfrac{\ln 3}{\ln x}$
$\to y' = \dfrac{-\ln 3 . \frac{1}{x}}{(\ln x)^2}$

$$f'(x) = 3^x \ln 3 + \ln 3 . \frac{-1}{x(\ln x)^2}$$

$$= \ln 3 \left[3^x - \frac{1}{x \ln^2 x} \right]$$

(e) Find $f'(1) = \dfrac{dy}{dx}\Big|_{x=1}$ if $y = f(x) = \dfrac{x^{9x}(x-2)^3}{x^4 e^x}$.

We can use logarithmic differentiation:

$\ln y = 9x \ln x + 3 \ln(x-2) - 4 \ln x - x$

$\Rightarrow \dfrac{1}{y} . y' = 9(\ln x + 1) + \dfrac{3}{x-2} - \dfrac{4}{x} - 1$

put $x = 1$ ($\Rightarrow y = \frac{-1}{e}$) to get

$$y'\Big|_{x=1} = \frac{-1}{e}(9 - 3 - 4 - 1) = \frac{-1}{e}$$

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Q.5 (12 pts) Verify that the point $P_0(\pi, 0)$ is on the curve C defined implicitly by the equation $\sin(x+y) = xy$. Then find the two <u>lines</u> which are **normal** and **tangent** to C at P_0.

$\left.\begin{array}{l} x=\pi \\ y=0 \end{array}\right\} \Rightarrow \quad \sin(\pi+0) \stackrel{?}{=} \pi \cdot 0$

$\sin\pi \stackrel{\checkmark}{=} 0$ so P_0 is on C

To find the slope of tangent line m at P_0 (and hence slope of normal line $-\frac{1}{m}$ at P_0), we need to find $y'\big|_{P_0}$.

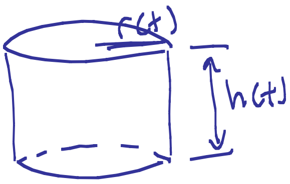
$\sin(x+y) = xy \Rightarrow \cos(x+y) \cdot (1+y') = y + xy'$

at P_0, we have $m = y'\big|_{P_0} = \dfrac{-1}{\pi+1}$ = slope of tangent line at P_0 to the curve C

Hence

- Equation of tangent line at P_0 is $y = \dfrac{-1}{\pi+1}(x-\pi)$
- " " normal " " " " $y = (\pi+1)(x-\pi)$

Q.6 (9 pts) The **radius** r of a right circular cylinder decreases at a rate of 0.3 cm/sec, and the **height** h increases at a rate of 0.2 cm/sec. Find the rate of change of the **volume** V of the cylinder when $r = 1$ cm and $V = 15\pi$ cm^3. Is V increasing or decreasing?



$r'(t) = -0.3$ cm/sec $\forall t$

$h'(t) = +0.2$ cm/sec $\forall t$

$V(t) = \pi r^2(t) h(t) \quad \forall t$

$\Rightarrow V'(t) = \pi \left[2r(t) r'(t) h(t) + r^2(t) \cdot h'(t) \right] \quad \forall t$

Let t_0 be the time when $r(t_0) = 1$ cm and $V(t) = 15\pi$ cm^3.

Then $h(t_0) = 15$ cm and hence

$V'(t_0) = \pi \left[2 \cdot 1 \cdot (-0.3) \cdot 15 + 1^2 \cdot (0{,}2) \right] = -8{,}8\pi$ cm^3/sec

Thus when $r(t) = 1$ cm and $V(t) = 15\pi$ cm^3, Volume is decreasing at a rate of $8{,}8\pi$ cm^3/sec.

M E T U
Department of Mathematics

	Calculus with Analytic Geometry Second Midterm Exam	
Code : MATH 119 Acad.Year : 2016 Semester : Fall Coord. : Muhiddin Uğuz Date : 24.12.2016 Time : 9.30 Duration : 119 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : 5 Questions on 4 Pages Total 100 Points	

Q1	Q2	Q3	Q4	Q5(a,b)	Q5(c,d)	SHOW YOUR WORK !	

Q.1 (2+5+5+8 = 20 pts) Given $f(x) = x^2 e^{-x^2}$ on $x \in [0, \infty)$, i.e. for $x \geq 0$.

(a) Find asymptotes if there is any.

H.A: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{e^{x^2}}$ $[\frac{\infty}{\infty}$ type$]$ $= \lim_{x\to\infty} \frac{\cancel{2x}}{\cancel{2x}\, e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \Rightarrow y=0$ is a Horizontal Asym.

since $f(x)$ is continuous on $[0,\infty)$, there is no vertical asymptote

(b) Find the critical points, and determine intervals of increase and decrease. Then calculate the local extreme values of f if there is any.

$f'(x) = \frac{2x\, e^{x^2} - 2x \cdot x^2 e^{x^2}}{(e^{x^2})^2} = \frac{2x(1-x^2)}{e^{x^2}}$ is defined $\forall x \in [0,\infty) = \mathrm{Dom}(f)$.

$f'(x) = 0 \Rightarrow x \in \{-1, 0, 1\}$. since $-1 \notin \mathrm{Dom}(f)$, $x_1 = 0$ & $x_2 = 1$ are the critical pts

x	0		1	
f'	0	$+$	0	$-$
f		$\nearrow$		$\searrow$

f is increasing on $[0,1]$

f is decreasing on $[1,\infty)$

f has local MAX at $x=1$, $f(1) = 1/e$

f has local min at $x=0$, $f(0) = 0$

(c) Determine intervals of concavity, and find the x–coordinates of inflection points if there is any.

$f'(x) = \frac{2(x-x^3)}{e^{x^2}} \Rightarrow f''(x) = 2\,[(1-3x^2)\cancel{e^{x^2}} - (x-x^3)2x\,\cancel{e^{x^2}}]\,/\,(e^{x^2})^{\cancel{2}}$

$= 2\,[1 - 3x^2 - 2x^2 + 2x^4]\,/\,e^{x^2}$

$= 2\,[2x^4 - 5x^2 + 1]/e^{x^2} = 2(2t^2 - 5t + 1)/e^t \quad (t = x^2)$

$f'' = 0 \Rightarrow 2t^2 - 5t + 1 = 0 \Rightarrow t = x^2 = (5 \mp \sqrt{25 - 4\cdot 2\cdot 1})/4 = \frac{5 \mp \sqrt{17}}{4}$

$\Rightarrow x = \mp\sqrt{\frac{5 \mp \sqrt{17}}{4}} > 0 \Rightarrow x_1 = \sqrt{\frac{5-\sqrt{17}}{4}}$, $x_2 = \sqrt{\frac{5+\sqrt{17}}{4}}$. x_1 & x_2 are inflection points

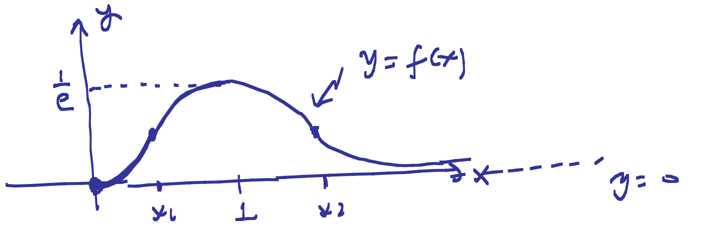
x	0		x_1		x_2	
f''		$+$	0	$-$	0	$+$
f		$\cup$		$\cap$		$\cup$

f is concave up on $[0, x_1]$ and on $[x_2, \infty)$

f is concave down on $[x_1, x_2]$

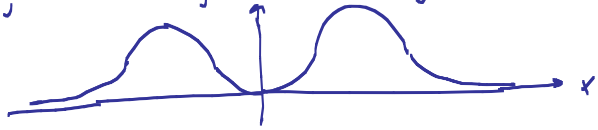
(d) Use the information above to make a TABLE showing the behaviour of f. Then sketch its graph. Without further calculation, can you sketch the graph on $x \in (-\infty, \infty)$?

x	0		x_1		1		x_2	
f'		$+$		$+$	0	$-$		$-$
f''		$+$	0	$-$		$-$	0	$+$
f		$\nearrow$ & $\cup$		$\nearrow$ & $\cap$		$\searrow$ & $\cap$		$\searrow$ & $\cup$



Note that on $\mathbb{R}$, $f(-x) = f(x)$, that is f is an even function on $\mathbb{R}$

thus on $\mathbb{R}$, graph of f is symmetric w.r.t. y-axis:



Q.2 **(4 × 5 = 20 pts)** Evaluate the limits (l'Hospital's rule may be used):

(a) $\lim_{x\to\infty} (\ln x)^{1/x}$

$= e^{\ln\left(\lim_{x\to\infty} (\ln x)^{1/x}\right)}$ $\overset{\ln x \text{ is cont.}}{=}$ $e^{\lim_{x\to\infty} \ln(\ln x)^{1/x}} = e^{\lim_{x\to\infty} \frac{\ln(\ln x)}{x}}$ $\left(\frac{\infty}{\infty} \text{ type}\right)$

$\overset{\text{l'Hopital Rule}}{=} e^{\lim_{x\to\infty} \frac{\frac{1/x}{\ln x}}{1}} = e^{\lim_{x\to\infty} \frac{1}{x\ln x}} = e^0 = 1$

(b) $\lim_{x\to 0} \dfrac{1-\cos\left(\frac{\sin^2 x}{x}\right)}{\left(\frac{\sin^2 x}{x}\right)^2}$

let $u = \frac{\sin^2 x}{x}$; $\lim_{x\to 0} u = \lim_{x\to 0} \overset{0}{\sin x} \overset{1}{\frac{\sin x}{x}} = 0.$

since $\cos u$ & u^2 are continuous.

$= \lim_{u\to 0} \frac{1-\cos u}{u^2}$ $\left(\frac{0}{0} \text{ type}\right)$

$= \lim_{u\to 0} \frac{\overset{1}{\sin u}}{2u} = \frac{1}{2}$

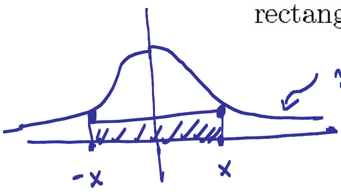
(c) $\lim_{x\to 1} \dfrac{2(x-1) - \int_x^1 e^{1-t^2}\,dt}{\int_1^{x^3} \cos(\pi t^2)\,dt}$

$\left(\frac{0}{0} \text{ type, since } \int_1^1 e^{1-t^2}dt = 0 = \int_1^1 \cos(\pi t^2)dt\right)$

$\overset{\text{l'Hopital Rule}}{=} \lim_{x\to 1} \frac{2-(-e^{1-x^2})}{\cos(\pi x^6)\cdot 3x^2} = \frac{2+e^0}{\cos(\pi).3} = \frac{2+1}{-3} = -1$

(d) $\lim_{x\to 0} \dfrac{\arcsin x + 3\ln(x+1)}{x + \sin x - \cos x} = \frac{\arcsin 0 + 3\ln(1)}{0+\sin 0 - \cos 0} = \frac{0+3\cdot 0}{0+0-1} = \frac{0}{-1} = 0$

Q.3 (12 pts) Consider the RECTANGLE with two vertices on the x-axis and the other two on the curve $y = f(x) = \mathrm{e}^{-x^2}$. Find the **largest** possible area for this rectangle. **Hint:** Note that $f(x)$ is EVEN, i.e. $f(x) = f(-x)$.

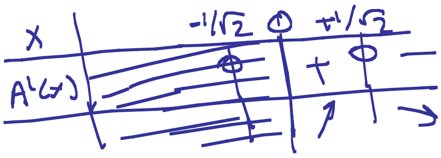


$y = f(x) = e^{-x^2}$

$\text{Area}(x) = A(x) = 2x \cdot e^{-x^2}$ Maximize $A(x)$ on $(0, \infty)$

$$A'(x) = 2e^{-x^2} + 2x(-2x)e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0$$

$$\Rightarrow x = \mp 1/\sqrt{2}$$



$\Rightarrow$ $A(x)$ has local max at $x = +1/\sqrt{2}$

Moreover $A(x)$ is continuous and

$$\lim_{x \to +\infty} A(x) = \lim_{x \to \infty} \frac{2x}{e^{x^2}} = 0$$

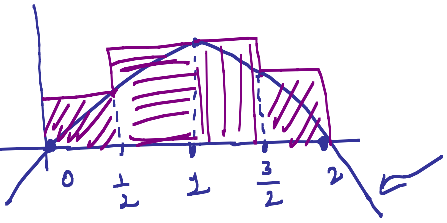
$$\lim_{x \to 0^+} A(x) = \lim_{x \to 0^+} \frac{2x}{e^{x^2}} = 0$$

Thus the local MAX $x = +1/\sqrt{2}$ is in fact global (Absolute) MAX and

$$A\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} e^{-1/2} = \frac{\sqrt{2}}{\sqrt{e}} = \sqrt{2/e}$$ is Absolute MAX value

Q.4 (8+12 = 20 pts)

(a) Write the UPPER Riemann sum for $f(x) = 2x - x^2$ on $x \in [0, 2]$ by dividing it into $n = 4$ subintervals of **equal** length.



$y = f(x) = x(2-x)$

$$\mathcal{U}(f, 4) = \Delta x \left(f\left(\tfrac{1}{2}\right) + f(1) + f(1) + f\left(\tfrac{3}{2}\right) \right) = \frac{1}{2}\left[\frac{3}{4} + 1 + 1 + \frac{3}{4}\right]$$

$$= \frac{7}{4}$$

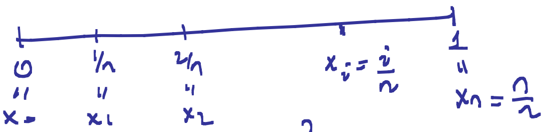
(b) Find $\lim_{n\to\infty} S_n$ if $S_n = \dfrac{1}{n}\left[\dfrac{\ln\left(1+\frac{1}{n}\right)}{1+\frac{1}{n}} + \dfrac{\ln\left(1+\frac{2}{n}\right)}{1+\frac{2}{n}} + \cdots + \dfrac{\ln\left(1+\frac{n-1}{n}\right)}{1+\frac{n-1}{n}} + \dfrac{\ln 2}{2}\right]$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{\ln(1+i/n)}{1+i/n}.$$

given limit is a Riemann sum for the integral $\int_0^1 \frac{\ln(1+x)}{1+x}\,dx$

(or for $\int_1^2 \frac{\ln x}{x}\,dx$, or for ....) where $\Delta x = \frac{1}{n}$

$\Delta x = \frac{1}{n}$,



$$\lim_{n\to\infty} S_n = \int_0^1 \frac{\ln(1+x)}{1+x}\,dx = \int_1^2 \frac{\ln u}{u}\,du \underset{\substack{t = \ln u \\ dt = \frac{1}{u}du}}{=} \int_0^{\ln 2} t\,dt = \frac{t^2}{2}\Big|_0^{\ln 2} = \frac{(\ln 2)^2}{2}$$

Q.5 (4 × 7 = 28 pts) Evaluate the following integrals:

(a) $\int x^3 e^{x^2}\,dx = I$

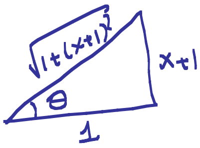
Let $t = x^2$, $dt = 2x\,dx$ $\Rightarrow$ $I = \frac{1}{2}\int \underbrace{t}_{u}\, \underbrace{e^t\,dt}_{dv} = \frac{1}{2}\left[te^t - \int e^t\,dt\right] = \frac{1}{2}\left[te^t - e^t\right] + c$

$du = dt$, $v = e^t$

$= \frac{1}{2} e^{x^2}(x^2 - 1) + c$

(OR $u = x^2$, $dv = x e^{x^2}\,dx$ $\Rightarrow$ ...)

(b) $\displaystyle\int_0^{\sqrt{3}-1} \frac{dx}{\sqrt{2+2x+x^2}} = \int_0^{\sqrt{3}-1} \frac{dx}{\sqrt{1+(x+1)^2}} = \int_{\theta_1}^{\theta_2} \cos\theta \cdot \sec^2\theta\,d\theta = \int_{\theta_1}^{\theta_2} \sec\theta\,d\theta$



$\cos\theta = \frac{1}{\sqrt{1+(x+1)^2}}$

$\tan\theta = x + 1$

$\sec^2\theta\,d\theta = dx$

$= \ln|\sec\theta + \tan\theta| \Big|_{\theta_1}^{\theta_2}$

$= \ln\left|\sqrt{1+(x+1)^2} + x + 1\right| \Big|_0^{\sqrt{3}-1} = \ln|\sqrt{4} + \sqrt{3}| - \ln|\sqrt{2} + 1|$

$= \ln \frac{2+\sqrt{3}}{1+\sqrt{2}}$

(c) $\int \sin^3 x \sqrt{\cos x}\,dx = \int \sin^2 x \cdot \sqrt{\cos x} \cdot \sin x\,dx = -\int (1-u^2)\, u^{1/2}\,du$

$u = \cos x$, $du = -\sin x\,dx$

$= \int \left(u^{5/2} - u^{1/2}\right) du = \frac{2u^{7/2}}{7} - \frac{2u^{3/2}}{3} + c$

$= \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + c$

(d) $\displaystyle\int \frac{14 - x - x^2}{(x+2)(x^2+2)}\,dx = I$

$\frac{14 - x - x^2}{(x+2)(x^2+2)} = \frac{A}{x+2} + \frac{Bx + C}{x^2+2}$ $\Rightarrow$ $A(x^2+2) + (Bx+C)(x+2) = 14 - x - x^2$

$x^2(A+B) + x(2B+C) + (2A+2C) = 14 - x - x^2$

$A + B = -1$, $2B + C = -1$, $A + C = 7$ $\Rightarrow$ $A = 2$, $B = -3$, $C = 5$

$I = 2\int \frac{1}{x+2}\,dx - 3\int \frac{x}{x^2+2}\,dx + 5\int \frac{1}{x^2+2}\,dx$

$= 2\ln|x+2| - \frac{3}{2}\ln(x^2+2) + \frac{5}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$

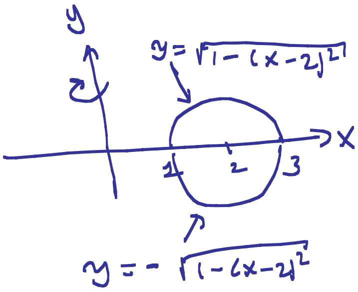
M E T U
Department of Mathematics

	Calculus with Analytic Geometry Final Exam	

Code : MATH 119 Acad.Year : 2016 Semester : Fall Coord. : Muhiddin Uğuz Date : 12.01.2017 Time : 9.30 Duration : 150 minutes	Last Name : Name : Stud. No : Dept. : Sec. No : Signature : **8** Questions on **6** Pages Total **100** Points

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	SHOW YOUR WORK!	

1.(10 pts) The circle $(x-2)^2 + y^2 = 1$ is rotated about the y-axis to obtain a surface T called a torus. Calculate the surface area of the torus T.



$$y^2 = 1-(x-2)^2 \Rightarrow y = \mp\sqrt{1-(x-2)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left[\frac{1}{2}\frac{-2(x-2)}{\sqrt{1-(x-2)^2}}\right]^2}\,dx$$

$$= \sqrt{1+\frac{(x-2)^2}{1-(x-2)^2}}\,dx = \frac{1}{\sqrt{1-(x-2)^2}}\,dx$$

Using the symmetry of the circle w.r.t. x-axis, we have:

$$\text{Surface Area} = SA = 2\int_1^3 2\pi x\sqrt{\frac{1}{\sqrt{1-(x-2)^2}}}\,dx.$$

Note that this is an improper integral (improper at $x=1$ & $x=3$) since it gives the volume of the torus (which is finite) we know that it is convergent. We'll treat it as a proper integral. Improper part will disappear as the function is odd in the next line. For more details, see the (*) below.

$$\left(\begin{array}{l} u = x-2 \to du = dx \\ x = u+2 \end{array}\right)$$

$$= 4\pi\int_{-1}^{1}\frac{u+2}{\sqrt{1-u^2}}\,du = 4\pi\Big[\underbrace{\int_{-1}^{1}\frac{u}{\sqrt{1-u^2}}\,du}_{=0} + 2\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\,du\Big]$$

odd function

$$= 8\pi(\arcsin u)\Big|_{-1}^{1}$$

$$= 8\pi\left[\arcsin 1 - \arcsin(-1)\right] = 8\pi\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = 8\pi^2$$

(*):

$$= 4\pi\int_{-1}^{1}\frac{u+2}{\sqrt{1-u^2}}\,du = 4\pi\left[\lim_{c\to -1^+}\int_c^0\left(\frac{u}{\sqrt{1-u^2}}+\frac{2}{\sqrt{1-u^2}}\right)du + \lim_{d\to 1^-}\int_0^d\left(\frac{u}{\sqrt{1-u^2}}+\frac{2}{\sqrt{1-u^2}}\right)du\right]$$

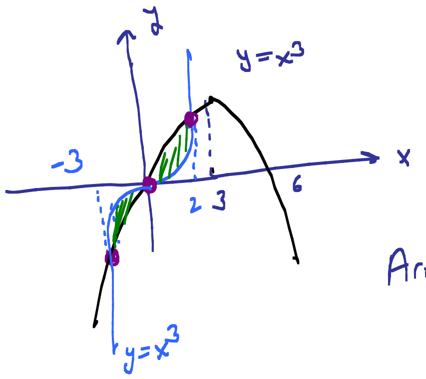
since each improper integral is convergent

$w = 1-u^2$, $dw = -2u\,du$

$$= 4\pi\left[\lim_{c\to -1^+}\left(\cancel{\int_{1-c^2}^{1}-\frac{1}{2}w^{-1/2}\,dw} + 2\arcsin u\Big|_c^0\right) + \lim_{d\to 1^-}\left(\cancel{\int_1^{1-d^2}-\frac{1}{2}w^{-1/2}\,dw} + 2\arcsin u\Big|_0^d\right)\right]$$

$$= 8\pi\Big[\underbrace{\arcsin 0}_{=0} - \underbrace{\arcsin(-1)}_{=-\pi/2} + \underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin 0}_{=0}\Big] = 8\pi\cdot\pi = 8\pi^2$$

2.(10 pts) Sketch the region D bounded by the curves $y = x^3$ and $y = 6x - x^2$. Then calculate the area of D.



$x^3 = 6x - x^2 \Rightarrow x^3 + x^2 - 6x = x(x^2+x-6)$

$= x(x+3)(x-2) = 0$

$x=-3$, $x=0$, $x=2$ are intersection pts

$\text{Area}(D) = A = \int_{-3}^{0} [x^3 - (6x - x^2)]\,dx + \int_0^2 [(6x-x^2) - x^3]\,dy$

$= \left(\frac{x^4}{4} + \frac{x^3}{3} - 3x^2\right)\Big|_{-3}^{0} + \left(3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2$

$= -\frac{81}{4} + 9 + 27 + 12 - \frac{8}{3} - 4 = 44 - \frac{8}{3} - \frac{81}{4}$

3.(5+5=10 pts)

(a) Find $\frac{dy}{dx}$ at $P(1,0)$ if $x + \sin(\frac{\pi x}{2} + y^2) + y = 2$.

Use implicit differentiation:

$\frac{d}{dx}\left[x + \sin\left(\frac{\pi x}{2} + y^2\right) + y\right] = \frac{d}{dx}[2] \Rightarrow 1 + \cos\left(\frac{\pi x}{2} + y^2\right)\cdot\left(\frac{\pi}{2} + 2yy'\right) + y' = 0$

At $(x,y) = (1,0)$ we have;

$1 + \cos\left(\frac{\pi}{2}\cdot 1 + 0^2\right)\left(\frac{\pi}{2} + 2\cdot 0\cdot y'\right) + y' = 0 \Rightarrow 1 + \cos\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2} + y' = 0$

$\Rightarrow 1 + y' = 0 \Rightarrow y'\big|_P = -1$

(b) Find $\frac{dy}{dx}$ if $y = (\sec x)^{\arcsin x}$.

$\ln y = \arcsin x \cdot \ln(\sec x)$. Differentiating both sides, we obtain

$\frac{y'}{y} = \frac{1}{\sqrt{1-x^2}} \ln(\sec x) + \arcsin x \cdot \frac{1}{\sec x} \cdot \sec x \tan x$

$\Rightarrow y' = (\sec x)^{\arcsin x} \left[\frac{1}{\sqrt{1-x^2}} \ln(\sec x) + \arcsin x \cdot \frac{1}{\sec x} \cdot \sec x \tan x\right]$

4. (6+6+6=18 pts)

(a) Is $\int_1^\infty \frac{\sin^2\left(\frac{1}{x}\right)}{\sqrt{x}}dx$ convergent or divergent? Explain.

Since $\lim_{x\to\infty} \frac{\sin 1/x}{1/x} = 1$, for large values of x, $\left(\sin\frac{1}{x}\right)^2 \approx \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$

Let $f(x) = \frac{\sin^2\left(\frac{1}{x}\right)}{\sqrt{x}} \geq 0$ and $g(x) = \frac{1}{x^2}\frac{1}{\sqrt{x}} = \frac{1}{x^{5/2}} \geq 0$. Both f & g are continuous on $[1,\infty)$

Moreover $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}}\right)^2 = 1^2 = 1$ a nonzero finite number. Thus by limit comparison test either both $\int_1^\infty f(x)dx$ & $\int_1^\infty g(x)dx$ converge or both diverge.

Since $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^{5/2}}dx$ is convergent by p-test ($p=\frac{5}{2}>1$), $\int_1^\infty f(x)\,dx$ is also convergent

(b) Is $\int_0^2 \frac{1}{\sqrt{x}(2-x)}dx$ convergent or divergent? Explain.

given improper integral is improper at $x=0$ & $x=2$.

Consider $\int_0^1 \frac{1}{\sqrt{x}(2-x)}dx$ & $\int_1^2 \frac{1}{\sqrt{x}(2-x)}dx$ separately. If both are convergent, so is the original integral. If one of them is divergent so is the original integral.

$\int_1^2 \frac{1}{2-x}dx \underset{\substack{u=2-x\\du=-dx}}{=} \int_0^1 \frac{1}{u}du$ divergent by p-test ($p=1\geq 1$)

$x\in(1,2) \Rightarrow f(x) = \frac{1}{\sqrt{x}(2-x)} > 0$, $g(x) = \frac{1}{2-x} > 0$ and both are continuous on $(1,2)$

Moreover $\lim_{x\to 2^-}\frac{f(x)}{g(x)} = \lim_{x\to 2^-}\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{2}}$. a nonzero finite number. Thus by LCT $\int_1^2 f(x)\,dx$ and hence $\int_0^2 f(x)\,dx$ is divergent

(c) Evaluate $\int_1^\infty \frac{1}{x^2(x+1)}dx$.

$f(x) = \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \Rightarrow 1 = Ax(x+1) + B(x+1) + Cx^2$

Thus by Partial Fraction Decomposition, we have

$x=0 \Rightarrow \boxed{B=1}$ $\quad x=1 \Rightarrow 1 = 2A+2B+C$

$x=-1 \Rightarrow \boxed{C=1}$ $\quad \Rightarrow \boxed{A=-1}$

$\int_1^\infty f(x)dx = \lim_{c\to\infty}\int_1^c\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx = \lim_{c\to\infty}\left(\ln\left|\frac{x+1}{x}\right| - \frac{1}{x}\right)\Big|_{x=1}^{x=c}$

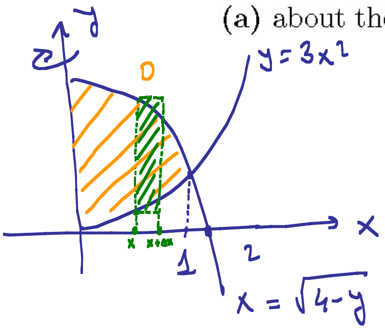
$= \lim_{c\to\infty}\left[\left(\ln\left(\frac{c+1}{c}\right) - \frac{1}{c}\right) - \left(\ln 2 - 1\right)\right] \underset{\ln x \text{ is continuous}}{=} \ln\left(\lim_{c\to\infty}\frac{c+1}{c}\right) - 0 - \ln 2 + 1$

$= 1 - \ln 2$. Thus given improper integral is convergent (converging to $1-\ln 2$)

5. **(6+6+6=18 pts)** Let D be the region with finite area which is bounded by the curves $y = 3x^2$, $x = \sqrt{4-y}$ and the y-axis.

Write down (but DO NOT EVALUATE) the definite integral(s) that represents the volume of the solid obtained by rotating the region D

(a) about the y-axis, using the cylindrical shells method.



$$x = \sqrt{4-y} \Rightarrow x^2 = 4-y \Rightarrow y = 4-x^2 \ \& \ x \geq 0$$

$$\left.\begin{array}{l} y = 3x^2 \\ y = 4-x^2 \end{array}\right\} \quad 3x^2 = 4-x^2 \Rightarrow x = \mp 1 \ \& \ x \geq 0$$

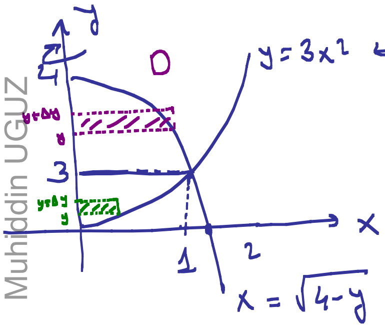
$\Rightarrow x = 1$ is the intersection point.

$$V = \int_0^1 2\pi r h \, dx = \underline{2\pi \int_0^1 x(4-4x^2)\,dx}$$

$r = x$

$h = 4-x^2-3x^2$

(b) about the y-axis, using the disk (slicing or washer) method.

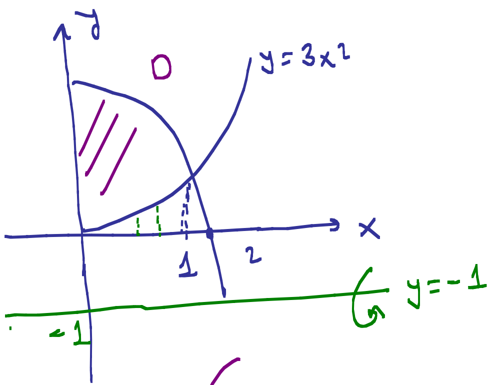


$y = 3x^2 \Leftrightarrow x = \sqrt{\frac{y}{3}}$

$$V = \int_0^3 \pi\left(\sqrt{\frac{y}{3}}\right)^2 dy + \int_3^4 \pi\left(\sqrt{4-y}\right)^2 dy$$

$$= \underline{\pi\left[\int_0^3 \frac{y}{3}\,dy + \int_3^4 (4-y)\,dy\right]}$$

(c) about the line $y = -1$.



Inner radius $r = 3x^2-(-1) = 3x^2+1$

Outher radius $R = 4-x^2-(-1) = 5-x^2$

$$V = \int_0^1 \pi(R^2-r^2)\,dx = \underline{\pi\int_0^1 \left[(5-x^2)^2-(3x^2+1)^2\right]dx}$$

$$\left(\text{OR: shells: } 2\pi\int_3^4 (y+1)\sqrt{4-y}\,dy + 2\pi\int_0^3 (y+1)\sqrt{\frac{y}{3}}\,dy\right)$$

NAME,SURNAME: *STUDENT ID:* *SIGNATURE:*

6. (4+8=12 pts) Consider the function

$$f(x)=\begin{cases} |x^2-1| & x\le 2\\ 2+\cos(x-2) & x>2\end{cases}$$

(a) Without finding the absolute extreme values, show that f has absolute maximum and minimum values on the interval $[0,6]$.

$[0,6]$ is a closed and finite (bounded) interval. If we show that $f(x)$ is continuous on $[0,6]$, then by Extreme Value Theorem, f has absolute MAX/min on $[0,6]$.

$|x^2-1|$ and $2+\cos(x-2)$ are continuous on $(-\infty,\infty)$; hence

$x_0\in[0,2) \Rightarrow \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} |x^2-1| = |x_0^2-1| = f(x_0)$

$x_1\in(2,6] \Rightarrow \lim\limits_{x\to x_1} f(x) = \lim\limits_{x\to x_1} 2+\cos(x-2) = 2+\cos(x_1-2) = f(x_1)$

$\Big\}$ f is continuous on $[0,2)\cup(2,6]$

$x=2 \Rightarrow \left[\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} |x^2-1| = 3,\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 2+\cos(x-2) = 3\right] \Rightarrow \lim\limits_{x\to 2} f(x) = 3 = f(2)$

$\therefore$ since $\lim\limits_{x\to a} f(x) = f(a)$ $\forall a\in[0,6]$, f is cont. on $[0,6]$.

(b) Find the absolute maximum and minimum values of f on the interval $[0,6]$.

Absolute Max/min of f on $[0,6]$, which exist by part (a), can occur at

(i) End pts: $\boxed{f(0)=1}$, $\boxed{f(6)=2+\cos 4}$

(ii) Critical pts: $f'(x)=0 \Rightarrow$
- on $(0,1)$ we have $f'(x)=-2x=0 \Rightarrow x=0\notin(0,1)$
- on $(1,2)$ " " $f'(x)=2x=0 \Rightarrow x=0\notin(1,2)$
- on $(2,6)$ we have $-\sin(x-2)=0 \Rightarrow x-2=k\pi$, $k\in\mathbb{Z}$ $\Rightarrow x=2+\pi$

$f(2+\pi) = 2+\cos\pi = \boxed{1}$

(iii) singular pts (f' d.n.e):

- $x=1$ is a singular pt in $(0,6)$ & $\boxed{f(1)=0}$
- f is defined by different formulas on the left and on the right of 2. Thus $x=2$ might be a singular pt. (indeed it is a singular pt i.e. $f'(2)$ does not exist), but without checking if $f'(2)$ exists or not, we can take $\boxed{f(2)=3}$ as a candidate for absolute MAX/min.

There are no other singular pts. ($f'(x)$ exists on $(0,6)\setminus\{1,2\}$)

Comparing all candidates for Abs. MAX/min in the boxes above, we obtain

$0 < 1 < 2+\cos(4) < 3$

Hence
- $f(1)=0$ is the absolute min value of f on $[0,6]$, $x=1$ is abs. min point
- $f(2)=3$ " " " MAX " ... , $x=2$ is abs. MAX pt.

7. (5+5=10 pts) Evaluate the following limits

(a) $\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+} e^{\sin x \cdot \ln x}$ $\overset{e^x \text{ is continuous}}{=}$ $e^{\lim\limits_{x\to 0^+} \sin x \cdot \ln x}$ $[0\cdot(-\infty) \text{ type}]$

$= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{1/\sin x}}$ $\left(\frac{\infty}{\infty}\text{ type}\right)$ $\overset{\text{L'Hôpital R.}}{=}$ $e^{\lim\limits_{x\to 0^+} \frac{1/x}{-\cos x/\sin^2 x}}$

$= e^{\lim\limits_{x\to 0^+}\left[\frac{\sin x}{x}\cdot\left(\frac{-\sin x}{\cos x}\right)\right]} = e^{1\cdot 0} = e^0 = 1$

(b) $\lim\limits_{x\to\infty} \frac{\sqrt{x^4+x+1}}{\sqrt[3]{x^6+1}} = \lim\limits_{x\to\infty} \frac{\cancel{x^2}\cdot\sqrt{1+\frac{1}{x^3}+\frac{1}{x^4}}}{\cancel{x^2}\cdot\sqrt[3]{1+\frac{1}{x^6}}}$ $\frac{1}{x^3}\to 0$, $\frac{1}{x^4}\to 0$, $\frac{1}{x^6}\to 0$ $= \frac{\sqrt{1+0+0}}{\sqrt[3]{1}} = 1$

8. (6+6=12 pts) Evaluate the following integrals

(a) $\int (x+2)\ln(x^2+2x+2)x$ $= \ln(x^2+2x+2)\left(\frac{x^2}{2}+2x\right) - \int \frac{\left(\frac{x^2}{2}+2x\right)(2x+2)}{(x+1)^2+1}dx$

$u = \ln(x^2+2x+2)$, $dv = (x+2)$

$du = \frac{2x+2}{x^2+2x+2}dx$, $v = \frac{x^2}{2}+2x$

$t = x+1 \to x = t-1$

$dt = dx$

$= \ln(x^2+2x+2)\left(\frac{x^2}{2}+2x\right) - \int \frac{(t-1)^2+4(t-1)}{t^2+1}\cdot t\, dt$

$= \ln(x^2+2x+2)\left(\frac{x^2}{2}+2x\right) - \int \frac{t^3+2t^2-3t}{t^2+1}dt = \ln(x^2+2x+2)\left(\frac{x^2}{2}+2x\right) - \underbrace{\int t+2 - \frac{4t+2}{t^2+1}dt}$

$\quad = \frac{t^2}{2}+2t-2\ln(t^2+1) - 2\arctan t + c$

$= \ln(x^2+2x+2)\left(\frac{x^2}{2}+2x\right) - \left[\frac{(x+1)^2}{2} + 2(x+1) - 2\ln(x^2+2x+2) - 2\arctan(x+1)\right] + C$

(b) $\int x^2\sqrt{1-x^2}x$ $= \int \sin^2\theta\cos\theta\cdot\cos\theta\, d\theta =$

$x = \sin\theta$

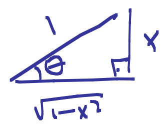
$dx = \cos\theta\, d\theta$

$\sqrt{1-x^2} = \cos\theta$

$= \int \sin^2\theta\cos^2\theta\, d\theta = \int(\sin\theta\cos\theta)^2 d\theta = \int\left(\frac{\sin(2\theta)}{2}\right)^2 d\theta = \frac{1}{4}\int \sin^2(2\theta)\, d\theta$

$= \frac{1}{4}\int \frac{1-\cos 4\theta}{2}d\theta = \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) + c$

$= \frac{1}{8}\left(\arcsin x - x\sqrt{1-x^2}\,(1-2x^2)\right) + C$



$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2\cdot 2\sin\theta\cos\theta(1-2\sin^2\theta)$

$= 4x\sqrt{1-x^2}\,(1-2x^2)$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	
Code : *Math 119* Acad. Year : *2015-2016* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *November.21.2015* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	SHOW YOUR WORK	

(6+6+6+6 pts) **1.** Evaluate each of the following limits if it exists. (DO NOT USE L'HOSPITAL'S RULE)

(a) $\lim_{x\to+\infty}\left(x-\sqrt{x^2-4x+1}\right)$ $\quad \infty-\infty$ type

$$= \lim_{x\to+\infty} \frac{\left(x-\sqrt{x^2-4x+1}\right)\left(x+\sqrt{x^2-4x+1}\right)}{\left(x+\sqrt{x^2-4x+1}\right)} = \lim_{x\to+\infty}\frac{\cancel{x^2}-\cancel{x^2}+4x-1}{x+\sqrt{x^2-4x+1}}$$

$x\to+\infty$ so $|x|=x$

$$= \lim_{x\to+\infty}\frac{\cancel{x}\left(4-\overset{\to 0}{\frac{1}{x}}\right)}{\cancel{x}\left(1+\sqrt{1-\underset{\to 0}{\frac{4}{x}}+\underset{\to 0}{\frac{1}{x^2}}}\right)} = \frac{4}{1+1} = 2$$

(b) $\lim_{x\to 0}\dfrac{3^{(2+x)^2}-81}{x}$ $= \lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = f'(0)$ where $f(x)=3^{(2+x)^2}$

so $\ln f(x) = (2+x)^2\ln 3$

$\frac{d}{dx}$ $\quad \frac{f'(x)}{f(x)} = 2(2+x)\ln 3$

$f'(x) = 3^{(2+x)^2}\cdot 2\cdot(2+x)\ln 3$

$$= f'(0) = 3^4\cdot 2\cdot 2\cdot \ln 3 = 324\ln 3$$

(c) $\lim_{x\to 0}\dfrac{\sin 5x}{\tan 7x}$ $= \lim_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\dfrac{5\cancel{x}\ \cos 7x}{\frac{\sin 7x}{7x}\cdot 7\cancel{x}}$

since each limit exists $\quad = \dfrac{5}{7}\lim_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\lim_{x\to 0}\cos 7x\cdot\dfrac{1}{\lim_{x\to 0}\frac{\sin 7x}{7x}} = \dfrac{5}{7}\cdot 1.1.1 = \dfrac{5}{7}$

(d) $\lim_{x\to+\infty}\dfrac{\sin(x^2+1)}{x^2}$

$-1 \le \sin(x^2+1) \le 1 \quad \forall x\in\mathbb{R}$

$\dfrac{-1}{x^2} \le \dfrac{\sin(x^2+1)}{x^2} \le \dfrac{1}{x^2} \quad \forall x>0$ & $\lim_{x\to+\infty}\mp\dfrac{1}{x^2}=0$

Therefore, by squeeze Theorem, $\lim_{x\to+\infty}\dfrac{\sin(x^2+1)}{x^2}=0$

(12 pts) **2.**

Find an equation of the tangent line to the curve $x^y = y^x$ at the point $(2,4)$.

$x^y = y^x \Rightarrow y\ln x = x\ln y \Rightarrow$ Take derivative of both sides with respect to x;

$$y'\ln x + y\frac{1}{x} = \ln y + x\frac{1}{y}\cdot y'$$

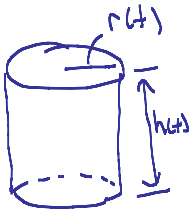
$$\Rightarrow y'(x,y) = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} \Rightarrow y'(2,4) = \frac{\ln 4 - \frac{4}{2}}{\ln 2 - \frac{2}{4}} = \frac{4\ln 2 - 4}{2\ln 2 - 1}$$

= slope of tangent line to the curve $x^y = y^x$ at $(2,4)$

Eqn. of tangent line:

$$\frac{y-4}{x-2} = \overbrace{\boxed{\frac{4\ln 2 - 4}{2\ln 2 - 1}}}^{m} \Rightarrow y = mx - 2m + 4 .$$

(12 pts) **3.** A cylinder is expanding so that its volume is increasing at a rate of $100\pi\,\mathrm{cm}^3$ /sec. If the radius of the cylinder is increasing at a rate of $\frac{1}{2}\,\mathrm{cm/sec}$ when the radius is $5\,\mathrm{cm}$ and height is $10\,\mathrm{cm}$, how fast is the **surface area** A of the cylinder changing at that moment?



$$V(t) = r^2(t)\,\pi\,h(t) \quad \forall t$$

$$\left.\frac{dV}{dt}\right|_t = 100\pi\ cm^3/sec \qquad \left.\frac{dr}{dt}\right|_{\substack{r(t)=5\ cm\\ h(t)=10\ cm}} = \frac{1}{2}\ cm/sec$$

$$A(t) = 2r^2(t)\pi + 2\pi r(t)h(t)$$

$$= 2\pi\left(r^2(t) + r(t)h(t)\frac{\pi}{\pi}\frac{r(t)}{r(t)}\right) = 2\pi\left(r^2(t) + \frac{V(t)}{\pi r(t)}\right)$$

$$A'(t) = 2\pi\left[2r(t)r'(t) + \frac{V'(t)\pi r(t) - V(t)\pi r'(t)}{\pi^2 r^2(t)}\right] \quad \forall t$$

Let $t = t_0$ denotes the specified moment. Then

$$A'(t_0) = 2\pi\left[2\cdot 5\cdot\frac{1}{2} + \frac{100\pi\cdot\pi\cdot 5 - \overbrace{V(t_0)}^{r^2(t_0)\pi h(t_0) = 5^2\cdot\pi\cdot 10 = 250\pi}\pi\frac{1}{2}}{\pi^2\cdot 25}\right]$$

$$= 2\pi\left[5 + \frac{\cancel{\pi}(500 - 125)}{25\cancel{\pi}}\right] = 2\pi\left[5 + \frac{375}{25}\right] = 2\pi(5+15)$$

$$= 40\pi\ cm^2/sec.$$

(14 pts) 4. Let $f(x) = \begin{cases} e^x(x^2+a) & \text{if } x<0, \\ 1 & \text{if } x=0, \\ bx^2+1 & \text{if } x>0. \end{cases}$

If possible, determine a, b so that $f'(0)$ exists. If not possible explain why?

$(f'(x) \text{ exists at } x=x_0) \Rightarrow (f(x) \text{ is continuous at } x=x_0)$

Check the continuity at $x_0=0$ $\quad$ ($f(x)$ is continuous $\forall x_0 \neq 0$, $\forall a, b \in \mathbb{R}$)

$$\left.\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} bx^2+1 = 1 \\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \underbrace{e^x}_{1}(x^2+a) = a \end{aligned}\right\} a=1 \quad (\Rightarrow \lim_{x\to 0} f(x) = f(0) = 1 \checkmark)$$

Now let find $f'(0)$:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{bh^2+1-1}{h} = \lim_{h\to 0^+} bh = 0 \quad \forall b$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{e^h(h^2+1)-1}{h} = \lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = g'(0)$$

$= 1 \quad \forall b$ $\qquad$ where $g(x) = e^x(x^2+1)$

$(\Rightarrow g'(x) = e^x(x^2+1) + e^x \cdot 2x$

$\Rightarrow g'(0) = 1)$

$a=1$ &

whatever b is chosen, $f'(0)$ does NOT exist.

$\therefore$ There is no such a, b.

(10 pts) 5. If $f:\mathbb{R}\to\mathbb{R}$ satisfies $f'(x) = e^{-x^2}$ for all x and $f(0)=0$ then prove that

$$|f(x)| \leq |x| \text{ for all } x.$$

$f'(x) = e^{-x^2}$ $\forall x \Rightarrow f(x)$ is continuous and differentiable on $\mathbb{R}$

For any $a \in \mathbb{R}\setminus\{0\}$,

if $a>0$ then $f(x)$ is continuous on $[0,a]$ and differentiable on $(0,a)$

if $a<0$ " " " " $[a,0]$ " " " $(a,0)$

Then by Mean Value Theorem

$$\frac{f(a)-f(0)}{a-0} = f'(c) \text{ for some } c \text{ between } 0 \text{ \& } a$$

(no matter if $a>0$ or $a<0$)

$$\Rightarrow \frac{f(a)}{a} = e^{-c^2} = \frac{1}{e^{c^2}} \qquad \left.\begin{aligned} &c^2>0 \\ &(c\neq 0) \\ &e^0=1 \end{aligned}\right\} \Rightarrow 0 < \frac{1}{e^{c^2}} < 1$$

$$\Rightarrow f(a) = a \cdot e^{-c^2} \Rightarrow |f(a)| = |a| \frac{1}{e^{c^2}} < |a|$$

$$\Rightarrow |f(a)| < |a| \text{ when } a\neq 0$$

$(a=0 \Rightarrow |f(0)| = |0| = 0)$

$\therefore$ $|f(x)| \leq |x|$ $\forall x \in \mathbb{R}$

(7+7+7+7 pts) 6.

(a) Find y' if $y = \dfrac{1}{\ln(\arcsin x)}$ $= [\ln(\arcsin x)]^{-1}$

$$y'(x) = -1 \cdot \frac{1}{[\ln(\arcsin x)]^2} \cdot \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}$$

(b) Find y' if $y = (x^2+119)^{\sqrt{x}}$ $\Rightarrow \ln y = \sqrt{x}\,\ln(x^2+119)$

$$\Rightarrow \frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln(x^2+119) + \sqrt{x}\,\frac{2x}{x^2+119}$$

$$\Rightarrow y' = (x^2+119)^{\sqrt{x}}\left[\frac{1}{2\sqrt{x}}\ln(x^2+119) + \frac{2x\sqrt{x}}{x^2+119}\right]$$

(c) Find $\dfrac{d}{dx}f^{-1}(x)$ at $x=2$ where $f(x) = x^3+5x+2$.

$$\underbrace{(f^{-1}\circ f)'(x)}_{(Id(x))' = x' = 1} = (f^{-1})'(f(x)) \cdot f'(x)$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

or equivalently

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)}$$

$$= \frac{1}{(3x^2+5)\big|_{x=0}} = \frac{1}{5}$$

(d) Let f and g be everywhere differentiable functions satisfying $f(1)=1$, $f'(1)=-2$, $f''(1)=4$ and $g(1)=0$, $g'(1)=3$, $g''(1)=-2$. Compute y'' at $x=1$ where $y = f(f(x)+2g(x))$.

$$y' = f'(f(x)+2g(x)) \cdot (f'(x)+2g'(x))$$

$$y'' = f''(f(x)+2g(x)) \cdot (f'(x)+2g'(x))^2 + f'(f(x)+2g(x))\,(f''(x)+2g''(x))$$

at $x=1$ we have

$$y'' = f''(1+2\cdot 0)\,((-2)+2\cdot 3)^2 + f'(1+2\cdot 0)\,\underbrace{(4+2(-2))}_{0}$$

$$= 4 \cdot 16 = 64$$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 2	
Code : *Math 119* Acad. Year : *2015-2016* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *26.12.2015* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	SHOW YOUR WORK	

(6+6+6 pts) **1.** Evaluate each of the following limits if it exists.

a) $\lim\limits_{x\to(\frac{\pi}{2})^-} \Big(\tan x\Big)^{\cos x}$ $\quad \infty^0$ type

$$e^{\ln \lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}} \overset{\ln x \text{ is continuous}}{=} e^{\lim\limits_{x\to\pi/2^-} \ln(\tan x)^{\cos x}} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} \cos x \ln(\tan x)} \quad (0\cdot\infty \text{ type})$$

$$= e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\ln(\tan x)}{1/\cos x}} \ \left(\frac{\infty}{\infty}\text{ type}\right) \quad 1/\cos x = \sec x \quad \overset{\text{L'Hopi's Rule}}{=} e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\frac{\sec^{\cancel{2}} x}{\tan x}}{\cancel{\sec x}\tan x}} = e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}}$$

$$= e^{\lim\limits_{x\to\frac{\pi}{2}^-} \frac{1}{\cancel{\cos x}}\frac{\cos^{\cancel{2}} x}{\sin^2 x}} = e^{\frac{0}{1}} = e^0 = 1$$

b) $\lim\limits_{x\to 0} \dfrac{e^{x^3}-1}{x^3\cos x}$ $\quad \left(\frac{0}{0}\text{ type}\right)$

$$\overset{\text{L'Hopi's Rule}}{=} \lim_{x\to 0} \frac{3\cancel{x^2}e^{x^3}}{3\cancel{x^2}\cos x - x^{\cancel{3}}\sin x} = \lim_{x\to 0}\frac{3e^{x^3}}{3\cos x - x\sin x} = \frac{3}{3} = 1$$

c) $\lim\limits_{n\to\infty} \dfrac{1}{n}\left(\sqrt{1-\dfrac{1}{n}}+\sqrt{1-\dfrac{2}{n}}+\cdots+\sqrt{1-\dfrac{n}{n}}\right) = \lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}\sqrt{1-i\frac{1}{n}}$

$0 \quad \frac{1}{n} \quad \frac{2}{n} \cdots \frac{i-1}{n} \quad \frac{i}{n} \quad \frac{i+1}{n} \cdots 1$

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}=1\}$ be partition of $[0,1]$ into n-equallength subintervals. Then $\Delta x_i = \frac{1}{n}$ and let $x_i^* = x_i$.
Take $f(x) = \sqrt{1-x}$ $\Rightarrow$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x_i f(x_i)$$

$$= \int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x}\,dx \qquad u = 1-x \to du = -dx$$

$$= \int_0^1 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = 2/3$$

(6+6 pts) **2.** Evaluate the following integrals.

a) $\displaystyle\int \frac{dx}{2^x+4^x}$ $= \displaystyle\int \frac{1}{u+u^2}\,\frac{1}{u.\ln 2}\,du = \frac{1}{\ln 2}\int \frac{1}{u^2(u+1)}\,du$

$u = 2^x$

$du = 2^x.\ln 2\, dx$

$\displaystyle\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \Rightarrow 1 = Au(u+1) + B(u+1) + Cu^2$

$\Rightarrow A + C = 0 \quad , \quad A + B = 0 \quad , \quad B = 1$

$\Rightarrow A = -1 \Rightarrow C = 1$

$\displaystyle = \frac{1}{\ln 2}\left[\int \frac{-1}{u}\,du + \int \frac{1}{u^2}\,du + \int \frac{1}{u+1}\,du\right] = \frac{1}{\ln 2}\left[-\ln|u| - \frac{1}{u} + \ln|u+1|\right] + C$

$\displaystyle = \frac{1}{\ln 2}\left[-\ln 2^x - \frac{1}{2^x} + \ln|2^x+1|\right] + C$

b) $\displaystyle\int_0^{\pi/3} x\tan^2 x\, dx$ $= \displaystyle\int_0^{\pi/3} x(\sec^2 x - 1)\,dx = x(\tan x - x)\Big|_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x)\,dx$

$u = x \rightarrow du = dx$

$dv = (\sec^2 x - 1)dx \rightarrow v = \tan x - x$

$\left(\displaystyle\int \tan x\, dx = \int \frac{\sin x}{\cos x}\,dx \quad u = \cos x \quad du = -\sin x\, dx \quad = -\ln|\cos x| + c\right)$

$\displaystyle = \frac{\pi}{3}\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$

$\displaystyle = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\left(\frac{1}{2}\right) + \frac{\pi^2}{18} = \frac{\pi}{\sqrt{3}} - \ln 2 - \frac{\pi^2}{18}$

(6+6 pts) **3.** Evaluate the following integrals.

a) $$\int \frac{x+2}{x^2+2x+2}\,dx = \int \frac{x+2}{1+(x+1)^2}\,dx \underset{\substack{u=x+1\\ du=dx}}{=} \int \frac{u+1}{1+u^2}\,du =$$

$$= \int \frac{u}{1+u^2}\,du + \int \frac{1}{1+u^2}\,du = \frac{1}{2}\int \frac{1}{w}\,dw + \arctan u + c$$

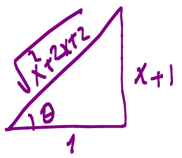
$w = 1+u^2$

$dw = 2u\,du$

$$= \frac{1}{2}\ln(1+u^2) + \arctan u + c$$

$$= \frac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) + c$$

or Let $x+1 = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

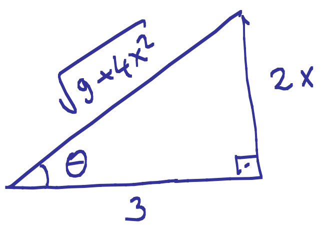


$$\Rightarrow \int \frac{x+2}{x^2+2x+2}\,dx = \int \frac{\tan\theta+1}{\cancel{1+\tan^2\theta}}\cancel{\sec^2\theta}\,d\theta = \int(\tan\theta+1)\,d\theta$$

$$= \ln|\cos\theta| + \theta + c = \ln\frac{1}{\sqrt{x^2+2x+2}} + \arctan(x+1) + c$$

$$= \frac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) + c$$

b) $$\int \frac{x^2}{(9+4x^2)^{5/2}}\,dx = \int \frac{\left(\frac{3}{2}\tan\theta\right)^2 \frac{3}{2}\sec^2\theta\, d\theta}{\left(\frac{3}{\cos\theta}\right)^5} = \frac{3^3}{2^3}\frac{1}{3^5}\int \frac{\sin^2\theta}{\cos^2\theta}\frac{1}{\cos^2\theta}\cos^5\theta\, d\theta$$



$\tan\theta = \frac{2x}{3}$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$

$\cos\theta = \frac{3}{\sqrt{9+4x^2}}$

$(9+4x^2)^{1/2} = \frac{3}{\cos\theta}$

$$= \frac{1}{72}\int \sin^2\theta\cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{72}\int u^2\,du$$

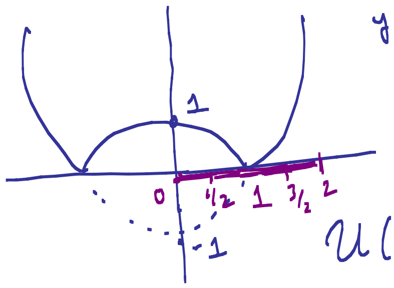
$$= \frac{1}{72}\frac{\sin^3\theta}{3} + c$$

$$= \frac{1}{216}\left[\frac{2x}{(9+4x^2)^{1/2}}\right]^3 + c$$

$$= \frac{x^3}{27(9+4x^2)^{3/2}} + c$$

(5+5 pts) **4.** Let $f(x) = |x^2 - 1|$.

a) Write down and compute the value of the upper Riemann sum for f for the partition of the interval $[0, 2]$ into 4 equal length subintervals.



$y = |x^2 - 1|$

$\Delta x_i = \frac{2-0}{4} = \frac{1}{2}$

$0 \quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2$

$\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

$U(f, \mathcal{P}) = \sum_{i=1}^{4} M_i \Delta x_i$ where M_i is the Maximum value of f on the i^{th} subinterval

$= f(0)\frac{1}{2} + f(\frac{1}{2})\frac{1}{2} + f(\frac{3}{2})\frac{1}{2} + f(2)\frac{1}{2}$

$= \left(1 + \frac{3}{4} + \frac{5}{4} + 3\right)\frac{1}{2} = 3$

b) Write down and compute the value of the lower Riemann sum for f for the partition of the interval $[0, 2]$ into 4 equal length subintervals.

$L(f, \mathcal{P}) = \sum_{i=1}^{4} m_i \cdot \Delta x_i$ where m_i is the minimum value of f on the ith subinterval

$= \frac{1}{2}\left(f(\frac{1}{2}) + f(1) + f(1) + f(\frac{3}{2})\right)$

$= \frac{1}{2}\left(\frac{3}{4} + 0 + 0 + \frac{5}{4}\right) = 1$

(12 pts) **5.** Let $F(x) = \int_1^{x^2} e^{t^2}\,dt$ be given. Using linear (tangent line) approximation of F, estimate $\int_1^{1.21} e^{t^2}\,dt$.

Note that $\int_1^{1.21} e^{t^2}\,dt = F(1.1)$

Linearization of F at 1 is $L(x) = F(1) + F'(1)(x-1)$

where $F(1) = \int_1^1 e^{t^2}\,dt = 0$, $\quad F'(x) = e^{x^4} \cdot 2x$

$F'(1) = 2e$

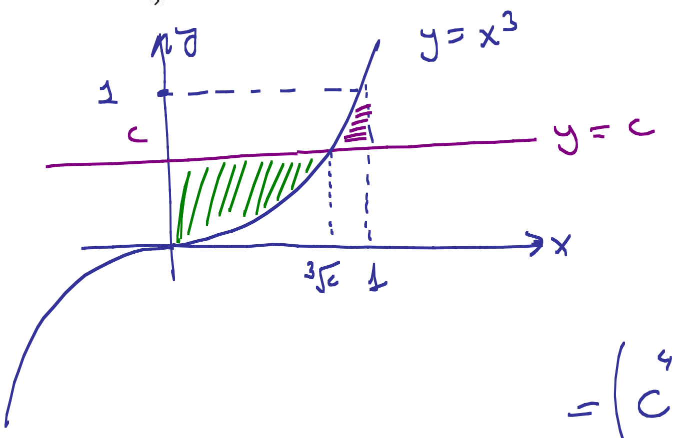
Thus $L(x) = 0 + 2e(x-1)$

$F(1.1) \approx L(1.1) = 0 + 2e(1.1 - 1) = \frac{e}{5}$

Thus $\int_1^{1.21} e^{t^2}\,dt \approx \frac{e}{5}$

(6+6 pts) 6. Let R be the region bounded by the curves $y = x^3, x = 0, x = 1$ and $y = c$ for a constant $0 \le c \le 1$.

a) Write down the total area of R in terms of the constant c.



$$\text{Area} = \int_0^{\sqrt[3]{c}} (c - x^3)\,dx + \int_{\sqrt[3]{c}}^{1} (x^3 - c)\,dx$$

$$= \left[cx - \frac{x^4}{4}\right]_{x=0}^{x=\sqrt[3]{c}} + \left[\frac{x^4}{4} - cx\right]_{x=\sqrt[3]{c}}^{1}$$

$$= \left(c^{4/3} - \frac{c^{4/3}}{4}\right) + \left(\frac{1}{4} - c\right) - \left(\frac{c^{4/3}}{4} - c^{4/3}\right)$$

$$= c^{4/3}\left(1 - \frac{1}{4} - \frac{1}{4} + 1\right) + \frac{1}{4} - c$$

$$= \frac{3}{2}c^{4/3} - c + \frac{1}{4}$$

b) Find the value of c so that the total area of R is minimum.

$$A(c) = \frac{3}{2}c^{4/3} - c + \frac{1}{4} \qquad 0 \le c \le 1$$

$A(c)$ is obviously continuous on the closed and bounded interval $[0,1]$. Thus $A(c)$ has MAX/min on $[0,1]$.

To find them;

check ① End points : $A(0) = \frac{1}{4}$, $A(1) = 3/4$

② Critical pts : $A'(c) = 2\sqrt[3]{c} - 1 = 0 \Rightarrow c = 1/8$

$$A\left(\tfrac{1}{8}\right) = \frac{3}{2}\left(\frac{1}{8}\right)^{4/3} - \frac{1}{8} + \frac{1}{4} = 7/32$$

Since A has minimum on $[0,1]$ and this minimum value must be one of $1/4$, $3/4$ & $7/32$, we can say that

$A(1/8) = 7/32$ is the minimum value. Hence minimum pt is $c = 1/8$

(3+3+6+6+6 pts) 7. Let $f(x) = (x-3)^2 e^x$.

a) Find the domain and intercepts of f

$\text{Domain}(f) = \mathbb{R} = (-\infty, \infty)$

x intercept : set $y = (x-3)^2 e^x = 0 \Rightarrow x = 3$ (x-intercept)

y intercept : set $x = 0 \Rightarrow y = f(0) = 9$ (y-intercept)

b) Find all vertical and horizontal asymptotes of the graph of f, if there are any. Explain.

Since f is continuous on $\mathbb{R}$, graph of f has no vertical asymptote

To find (if any) horizontal Asymptote: $\lim\limits_{x\to+\infty} (x-3)^2 e^x = +\infty$ no HA on the right.

$$\lim_{x\to-\infty} (x-3)^2 e^x = \lim_{x\to-\infty} \frac{(x-3)^2}{e^{-x}} = \lim_{x\to-\infty} \frac{2(x-3)}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

$y = 0$ is HA on the left

c) Find the intervals where f is increasing/decreasing.

$$f'(x) = 2(x-3)e^x + (x-3)^2 e^x = e^x (x-3)(2 + (x-3))$$
$$= e^x (x-3)(x-1) = 0 \Rightarrow x = 3 \ \& \ x = 1 \text{ are critical pts.}$$

x		1		3	
$f'(x)$	+	0	−	0	+
f	↗		↘		↗

f is increasing on $(-\infty, 1]$ and $[3, \infty)$

f is decreasing on $[1, 3]$

d) Find the intervals where f is concave up/down and all inflection points of f.

$$f'(x) = (x^2 - 4x + 3)e^x \Rightarrow f''(x) = (2x-4)e^x + (x^2-4x+3)e^x$$
$$= e^x (x^2 - 2x - 1)$$

$$f''(x) = 0 \Rightarrow x = \frac{2 \mp \sqrt{8}}{2} = 1 \mp \sqrt{2}$$

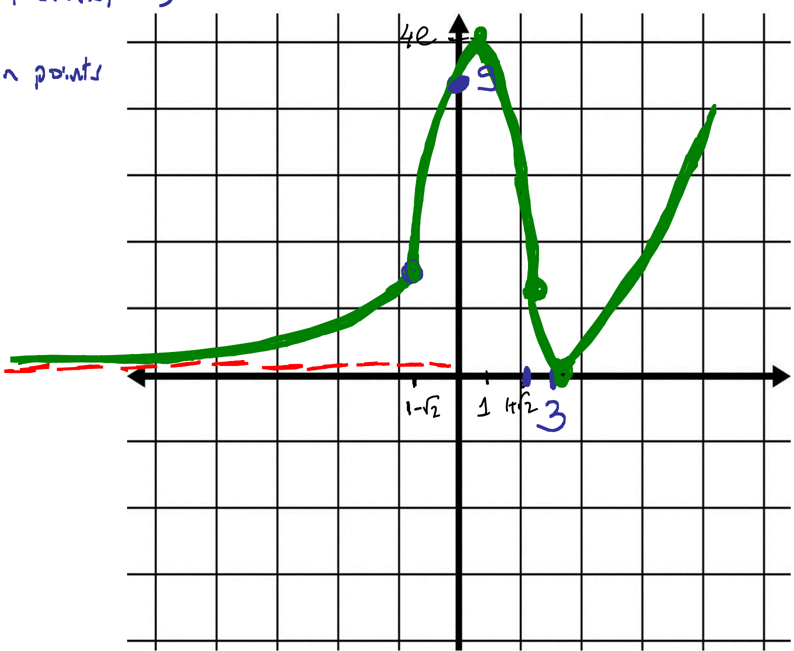
$\Rightarrow f$ is concave up on $(-\infty, 1-\sqrt{2})$ & $(1+\sqrt{2}, +\infty)$

" " " down " $(1-\sqrt{2}, 1+\sqrt{2})$

and $x = 1-\sqrt{2}$, $x = 1+\sqrt{2}$ are inflection points

x		$1-\sqrt{2}$		1		$1+\sqrt{2}$		3	
$f'(x)$	+		+	0	−		−	0	+
f''	+	0	−		−	0	+		+
f	↗		↗		↘		↘		↗
f	∪		∩		∩		∪		∪
f									

e) Sketch the graph of f.



M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY Final	

Code : *Math 119* Acad. Year : *2015-2016* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *14.01.2016* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature :
	8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

***(10 pts)* 1.** Let $f(x) = \begin{cases} ax+b & \text{if } x<0, \\ \sin x & \text{if } x=0, \\ \tan 2x & \text{if } x>0. \end{cases}$

Find all values of a and b, if they exist, so that $f'(0)$ exists.

$f'(0)$ exists $\Rightarrow$ ($f(x)$ is continuous at $x=0$) $\Leftrightarrow \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0) = \sin 0 = 0$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} ax+b = \boxed{b=0}$$

$$f'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\tan 2x - 0}{x} = \lim_{x\to 0^+} \overset{1}{\frac{\sin 2x}{2x}} \cdot \overset{2}{\frac{2}{\cos 2x}} = 2$$

$$= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{ax-0}{x} = a$$

$$\boxed{a=2}$$

***(5+5 pts)* 2.** Evaluate:

a) $\dfrac{dy}{dx}$ where $y = \tan(sec^2(x^2))$

By chain rule $\dfrac{dy}{dx} = \sec^2(\sec^2(x^2)) \cdot 2\sec(x^2)\cdot\sec(x^2)\cdot\tan(x^2)\cdot 2x$

b) $\lim\limits_{x\to 0+} \sqrt{x}\ln x$ $(0\cdot(-\infty)$ type$)$

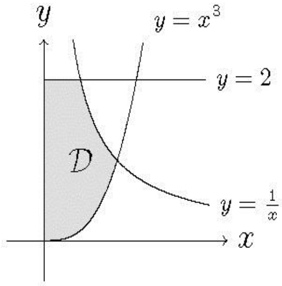
$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} \quad \left(\frac{\infty}{\infty} \text{ type}\right) \overset{\text{L'Hopital R.}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} (-2)\sqrt{x} = 0$$

(6+6+6 pts) 3.

Let $\mathcal{D}$ be the region in the plane described by all four inequalities

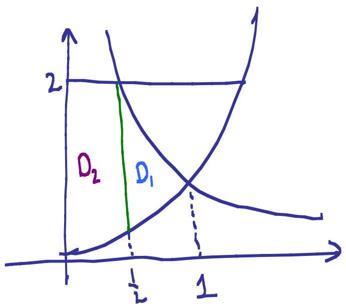
$$x \geq 0, \quad y \leq 2, \quad y \geq x^3, \quad y \leq \frac{1}{x}$$

(see the figure on the right).



a) Let $\mathcal{S}$ be the solid of revolution obtained by rotating the region $\mathcal{D}$ about the y-axis.

(i) Set up the integral(s) which give the volume of $\mathcal{S}$ using cylindrical shells method (DO NOT EVALUATE).



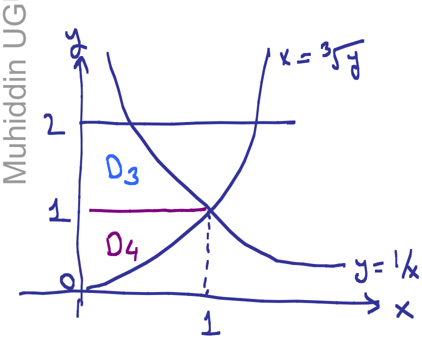
Divide D into two regions D_1 and D_2

$V = \int_a^b 2\pi r h \, dx$ (cylindrical shells mtd) where $r = x$ for both D_1 & D_2

Thus

$$V = \int_0^{1/2} 2\pi x (2 - x^3) dx + \int_{1/2}^{1} 2\pi x \left(\frac{1}{x} - x^3\right) dx$$

(ii) Set up the integral(s) which give the volume of $\mathcal{S}$ using washer (slicing, disc) method (DO NOT EVALUATE).



Washer mtd: $V = \int_a^b \pi (R^2 - r^2) dx$ r = inner radius, R = outher radius

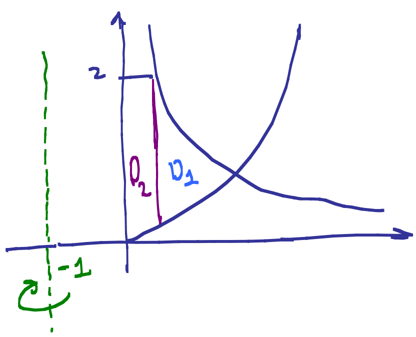
In D_3 : $r = 0$, $R = 1/y$

In D_4 : $r = 0$, $R = \sqrt[3]{y}$

$$V = \int_0^1 \pi (\sqrt[3]{y})^2 dy + \int_1^2 \pi \left(\frac{1}{y}\right)^2 dy$$

b) Let $\mathcal{T}$ be the solid of revolution obtained by rotating the region $\mathcal{D}$ about the line $x = -1$. Set up but (DO NOT EVALUATE) the integral(s) which give the volume of $\mathcal{T}$ using cylindrical shells method.

Cylindrical shells mtd: $V = \int_a^b 2\pi r h \, dx$



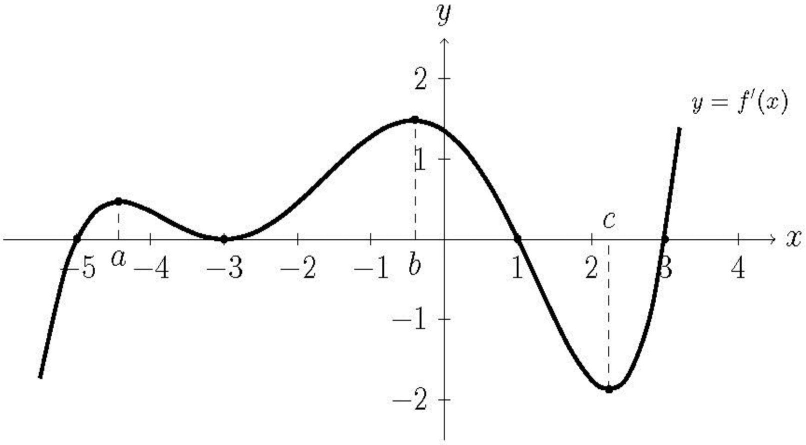
$r = x - (-1) = x + 1$

$h = \left(\frac{1}{x} - x^3\right)$ in D_1

$h = (2 - x^3)$ in D_2

$$V = \int_{1/2}^{1} 2\pi (x+1) \left(\frac{1}{x} - x^3\right) dx + \int_0^{1/2} 2\pi (x+1)(2 - x^3) dx$$

(3+3+3+3 pts) **4.** The graph of the derivative $f'(x)$ of a function $f(x)$ is given.



a) Determine the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

$(f'(x) \geq 0 \Rightarrow f$ is increasing$) \Rightarrow f$ is increasing on $[-5, 1]$ and on $[3, \infty)$

$(f'(x) \leq 0 \Rightarrow f$ is decreasing$) \Rightarrow f$ is decreasing on $(-\infty, -5]$ and on $[1, 3]$

b) Find and classify points of local extrema values of $f(x)$.

$f'(x) = 0 \Rightarrow x \in \{-5, -3, 1, 3\}$ is the set of critical pts. (no singular or no end points)

x		-5		-3		1		3	
f'	−	0	+	0	+	0	−	0	+
f	↘		↗		↗		↘		↗

- local minimum points of f are $x = -5$ & $x = 3$
- " MAX " " " $x = 1$
- $x = -3$ is a critical pt which is neither loc. max nor loc. min (it is an inflection pt)

c) Determine the intervals where the graph of $y = f(x)$ is concave up and the intervals where the graph of $y = f(x)$ is concave down.

$(f'(x)$ increasing $\Rightarrow f$ concave up$)$: f is concave up on $(-\infty, a]$, on $[-3, b]$, on $[c, \infty)$

$(f'(x)$ decreasing $\Rightarrow f$ concave down$)$: f is concave down on $[a, -3]$, on $[b, c]$

d) Determine inflection points of the graph of $y = f(x)$.

f' changes from ↗ to ↘ or ↘ to ↗

$x = x_0$ is an inflection pt. for $f(x)$ if tangent line at $(x_0, f(x_0))$ exists and concavity changes

$\Rightarrow$ inflection pts are $(a, f(a))$, $(-3, f(-3))$, $(b, f(b))$ & $(c, f(c))$

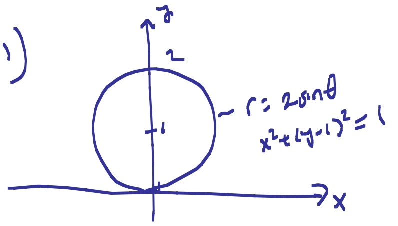
(4 pts) **5.** A curve is given by the polar equation $r = 2\sin\theta$. Convert this equation into cartesian coordinates (xy-coordinates) and sketch its graph.

$r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y + 1 = 1$
$(r \neq 0)$

$(r = 0 \Rightarrow (x, y) = (0, 0))$

$\Rightarrow x^2 + (y-1)^2 = 1$

circle centered at $(0, 1)$ of radius 1.



(6+6+6 pts) 6.

a) Determine whether the following integrals are convergent or divergent.

(i) $\int_0^\infty \frac{1+\sin^2\sqrt{x}}{x+x^4}dx = \int_0^\infty f(x)dx$ is improper at $x=0$ & ∞.

Consider $\int_0^1 f(x)dx$ & $\int_1^\infty f(x)dx$ seperately. If each one is convergent then $\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx$ is convergent. If any one is divergent, so is $\int_0^\infty f(x)dx$.

Consider $\int_0^1 f(x)dx$

$x \in [0,1] \Rightarrow 1+\sin^2 x \geq 1$

$0 < x+x^4 \leq x+x = 2x$

$\Rightarrow 0 < \frac{1}{2x} \leq \frac{1}{x+x^4}$

$\int_0^1 \frac{1+\sin^2\sqrt{x}}{x+x^4}dx \geq \underbrace{\int_0^1 \frac{1}{2x}dx}_{\text{divergent by (p-test) } (p=1)} \geq 0$

divergent by Comparison test

Hence $\int_0^\infty f(x)dx$ is also divergent

(ii) $\int_0^1 \frac{\sin(\frac{1}{x})}{\sqrt{x}}dx = \int_0^1 f(x)dx$ is improper at $x=0$

Consider $\int_0^1 \overbrace{|f(x)|}^{\geq 0} dx$ instead

$\Rightarrow \left(0 \leq \frac{|\sin\frac{1}{x}|}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}\right)$ and since $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent by p test $(p=\frac{1}{2})$

by comparison test $\int_0^1 |f(x)|dx$ is also convergent

Hence $\int_0^1 f(x)dx$ is absolutely convergent, so convergent.

b) Evaluate $\int_1^\infty \frac{\ln x}{x^2}dx$

$\left(\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + c\right)$

$u = \ln x$, $dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx$, $v = \frac{-1}{x}$

Thus

$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{R\to\infty}\int_1^R \frac{\ln x}{x^2}dx = \lim_{R\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^R = \lim_{R\to\infty}\left(-\frac{\ln R}{R} - \frac{1}{R} + 1\right)$

since each limit exists $\Rightarrow = -\left[\underbrace{\lim_{R\to\infty}\frac{\ln R}{R}}_{\frac{\infty}{\infty}\text{ type}} + \underbrace{\lim_{R\to\infty}\frac{1}{R}}_{=0} - \underbrace{1}_{=1}\right] = +1$

$= \lim_{R\to\infty}\frac{1}{R} = 0$

NAME: .... SURNAME: .... ID: .... SIGNATURE: .....

***(6+6+6 pts)* 7.** Evaluate the following integrals.

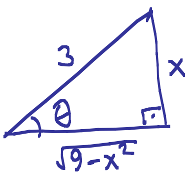
a) $\displaystyle\int \frac{dx}{e^x - 1}$ $= \displaystyle\int \frac{e^x\,dx}{e^x(e^x-1)} = \int \frac{du}{u(u-1)} = \int -\frac{1}{u} + \frac{1}{u-1}\,du$

$u = e^x$
$du = e^x dx$

$\dfrac{1}{u(u-1)} = \dfrac{A}{u}_{(u-1)} + \dfrac{B}{u-1}_{(u)} = \dfrac{u(A+B)-A}{u(u-1)} \Rightarrow \begin{matrix} A=-1 \\ B=1 \end{matrix}$

$= \ln\left(\dfrac{u-1}{u}\right) + C = \ln\left(\dfrac{e^x-1}{e^x}\right) + C$

b) $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$ $= \displaystyle\int \frac{\cancel{3\cos\theta}\,d\theta}{9\sin^2\theta\,\cancel{3\cos\theta}} = \frac{1}{9}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \csc^2\theta\,d\theta$



$\sin\theta = \dfrac{x}{3}$

$\cos\theta\,d\theta = \dfrac{1}{3}dx$

$\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$

$= -\dfrac{1}{9}\cot\theta + C$

$= -\dfrac{1}{9}\,\dfrac{\sqrt{9-x^2}}{x} + C$

c) $\displaystyle\int_1^{5/2} \frac{dx}{x\sqrt{2x-1}}$ $= \displaystyle\int_1^2 \frac{\cancel{u}\,du}{\left(\frac{1+u^2}{2}\right)\cancel{u}} = 2\int_1^2 \frac{du}{1+u^2} = 2\arctan u\Big|_1^2$

$u^2 = 2x-1 \Rightarrow u = \sqrt{2x-1}$

$\cancel{2}u\,du = \cancel{2}dx$

$x = 1 \Rightarrow u = 1$

$x = \frac{5}{2} \Rightarrow u = 2$

$= 2\left(\arctan(2) - \dfrac{\pi}{4}\right)$

(10 pts) 8. Find the absolute maximum and absolute minimum (if they exist) of

$$f(x) = e^{-\left(\frac{1+x^2}{x}\right)} \text{ on } (0,\infty).$$

$$\lim_{x\to 0^+} e^{-\left(\frac{1+x^2}{x}\right)} = e^{-\boxed{\lim_{x\to 0^+}\left(\frac{1+x^2}{x}\right)}^{=\infty}} = 0$$

$$\lim_{x\to\infty} e^{-\left(\frac{1+x^2}{x}\right)} = e^{-\lim_{x\to\infty}\frac{1+x^2}{x}} = e^{-\lim_{x\to\infty} 2x} = 0$$

$$f'(x) = e^{-\left(\frac{1+x^2}{x}\right)} \cdot \left(\frac{1}{x^2} - 1\right)$$ exists $\forall x > 0$: no singular points.

critical pts : $f'(x) = 0 \Rightarrow \frac{1}{x^2} = 1 \Rightarrow x = \mp 1 \Rightarrow x = +1$ is the only critical point in $(0,\infty)$

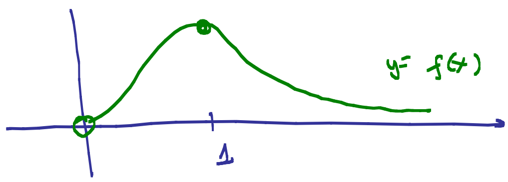
x	0		1	
$f'(x)$		$+$	0	$-$
$f(x)$	0	$\nearrow$		$\searrow$ 0

- f is increasing on $(0,1]$ and decreasing on $[1,\infty)$

$\Rightarrow$ Absolute MAX is at $x = 1$

- $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to\infty} f(x)$ & $f(x) > 0 \ \forall x > 0$

$\Rightarrow$ There is NO abs. min on $(0,\infty)$



(a rough sketch)

M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	Mid Term I	04.04.2015	13:30
Last Name: Name : Student No:	Signature : Duration : 100 *minutes*			
6 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS	
1 | 2 | 3 | 4 | 5 | 6	SHOW YOUR WORK			

1. (5+5+7+8 pts) This question has unrelated parts about derivatives.

(a) Evaluate $\frac{d}{dx}\left[\sin^2(2x)+\frac{e^x}{\ln x}\right]$

$$\frac{d}{dx}\left[\sin^2(2x)+\frac{e^x}{\ln x}\right] = 2\sin(2x)\cdot\cos(2x)\cdot 2 + \frac{e^x\ln x - \frac{e^x}{x}}{(\ln x)^2}$$

$$= 2\sin 4x + \frac{e^x\,(x\ln x - 1)}{x(\ln x)^2}$$

(b) Evaluate $\frac{d}{dx}\left[\ln\left(\frac{\sin x}{\sqrt[3]{2x+x^2}}\right)\right]$

$$\left[\text{OR: } \frac{d}{dx}\left[\ln\left(\frac{\sin x}{\sqrt[3]{2x+x^2}}\right)\right] = \frac{\dfrac{\cos x\sqrt[3]{2x+x^2} - \frac{1}{3}(2x+x^2)^{-2/3}(2+2x)\sin x}{(2x+x^2)^{2/3}}}{\dfrac{\sin x}{(2x+x^2)^{1/3}}}\right]$$

$$\frac{d}{dx}\left[\ln\left(\frac{\sin x}{\sqrt[3]{2x+x^2}}\right)\right] = \frac{d}{dx}\left[\ln\sin x - \frac{1}{3}\ln(2x+x^2)\right]$$

$$= \frac{\cos x}{\sin x} - \frac{1}{3}\,\frac{2+2x}{2x+x^2}$$

(c) Suppose that $h(x) = f(x^2+g(x))$, where $f(3)=6$, $f'(3)=-4$, $f(-2)=0$, $f'(-2)=1$, $g(3)=1$, $g'(3)=-1$, $g(-2)=-1$ and $g'(-2)=2$. Compute $h'(-2)$.

$$h(x) = f(x^2+g(x)) \Leftarrow\!\Rightarrow h'(x) = f'(x^2+g(x))\,(2x+g'(x))$$

$$h'(-2) = f'(4+g(-2))\,(-4+g'(-2))$$

$$h'(-2) = f'(3)(-2)$$

$$h'(-2) = (-4)(-2) = 8 \Leftarrow\!\Rightarrow \boxed{h'(-2)=8}$$

(d) Find y' at $(1,1)$ if $x^3+y^3+x^2y = \cancel{19}\ 3$. 3 (correction)

$x^3+y^3+x^2y=3$ Differentiate wrt x

$3x^2+3y^2\cdot y' + 2xy + x^2y' = 0$ Insert $x=1$ and $y=1$,

$3+3y'(1)+2+y'(1)=0$

$4y'(1) = -5$

$$\boxed{y'(1) = -\frac{5}{4}}$$

3.(12 pts) Suppose that f is a function for which $f(\pi/2) = 2$ and $f'(\pi/2) = 1$. Let

$$g(x) = f(x)^{\cos x}$$

Find an equation of the line tangent to the graph of $g(x)$ at $x = \pi/2$.

$$g'(x) = \left(e^{\cos x \ln f(x)}\right)' = e^{\cos x \ln(f(x))} \cdot \left[-\sin x \ln f(x) + \cos x \frac{f'(x)}{f(x)}\right]$$

$$g'\left(\tfrac{\pi}{2}\right) = f\left(\tfrac{\pi}{2}\right)^{\cos\frac{\pi}{2}} \left[-\sin\tfrac{\pi}{2} \ln f\left(\tfrac{\pi}{2}\right) + \cos\tfrac{\pi}{2} \frac{f'(\pi/2)}{f(\pi/2)}\right]$$

$g'\left(\tfrac{\pi}{2}\right) = 2^0 \left[-\ln 2\right] = -\ln 2$ is the slope of the tangent line at $x = \frac{\pi}{2}$

At $x = \frac{\pi}{2} \Rightarrow g\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)^{\cos\frac{\pi}{2}} = 2^0 = 1$. Thus, the tangent line eqn. to $g(x)$ at $x = \frac{\pi}{2}$ is:

$$y - 1 = -\ln 2 \left(x - \tfrac{\pi}{2}\right)$$

$$\boxed{y = -\ln 2\, x + \tfrac{\pi}{2}\ln 2 + 1}$$

4.(13pts) Show that

$$f(x) = 1 + \arctan x + e^x$$

has an inverse function. Find the derivative of the inverse of $f(x)$ at $x = 2$, that is, find $(f^{-1})'(2)$.

$f(x) = 1 + \arctan x + e^x$

$f'(x) = \frac{1}{1+x^2} + e^x > 0$ for all x. Thus $f(x)$ is increasing, hence 1-1 function.

Therefore, it has inverse, say $f^{-1}(x)$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

let $a = f^{-1}(2) \Leftrightarrow f(a) = 2$

$1 + \arctan a + e^a = 2 \Leftrightarrow \boxed{a = 0} = f^{-1}(2)$

Then $(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$

$$\boxed{(f^{-1})'(2) = \tfrac{1}{2}}$$

Question 2. (5+5+5+5+8 pts) This question has unrelated parts about limit.

(a) Evaluate $\lim_{x\to 3} \frac{|x^2-9|}{\sqrt[3]{x-3}}$

$$\lim_{x\to 3^+} \frac{x^2-9}{\sqrt[3]{x-3}} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{(x-3)^{1/3}} = \lim_{x\to 3^+} (x-3)^{2/3}(x+3) = 0$$

$$\lim_{x\to 3^-} \frac{-(x^2-9)}{\sqrt[3]{x-3}} = \lim_{x\to 3^-} \frac{-(x-3)(x+3)}{(x-3)^{1/3}} = \lim_{x\to 3^-} -(x-3)^{2/3}(x+3) = 0$$

Thus $\boxed{\lim_{x\to 3} \frac{|x^2-9|}{\sqrt[3]{x-3}} = 0}$

(b) Evaluate $\lim_{x\to\infty} (x - \sqrt{x^2+x+2})$

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+x+2}\right)^{(\infty-\infty)} = \lim_{x\to\infty} \left(\frac{\cancel{x^2} - \cancel{x^2} - x - 2}{x + \sqrt{x^2+x+2}}\right) = \lim_{x\to\infty} \frac{-\cancel{x}\left(1+\frac{2}{x}\right)}{\cancel{x}\left(1+\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}\right)}$$

$$= \frac{-1}{2}$$

(c) Evaluate $\lim_{x\to\frac{\pi}{2}^-} \frac{e^x+3x+1}{\sin x - 1}$ $\longrightarrow$ $-\infty$ because $\lim_{x\to\frac{\pi}{2}^-} (e^x+3x+1) = e^{\pi/2} + \frac{3\pi}{2} + 1$ exists,

but $\lim_{x\to\frac{\pi}{2}^-} (\sin x - 1) = 0$

(d) Find **all** values of a and b so that $\lim_{x\to\infty} \left[(ax+b) - \sqrt{x^2-x+\frac{1}{4}}\right]$ exists.

$$\lim_{x\to\infty} \left[(ax+b) - \sqrt{\left(x-\tfrac{1}{2}\right)^2}\right] = \lim_{x\to\infty} \left[ax+b-\left|x-\tfrac{1}{2}\right|\right] = \lim_{x\to\infty} \left[(a-1)x + \left(b+\tfrac{1}{2}\right)\right]$$

as $x\to\infty$, $x-\frac{1}{2} > 0$

limit exists iff $a-1=0$, i.e. $a=1$ and b is any finite numbers.

(e) Consider $f(x)$ is a continuous function for all x. Find a, where

$$f(x) = \begin{cases} 2x + (x-1)\sin\frac{1}{x-1} & x < 1 \\ ax+2 & x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(2x + (x-1)\sin\frac{1}{x-1}\right) = 2 \quad \text{since}$$

$\lim_{x\to 1^-} (x-1)\sin\frac{1}{x-1} = 0$ by Squeeze Thm (

That is, $\left|(x-1)\sin\frac{1}{x-1}\right| \leq |x-1| \iff -|x-1| \leq (x-1)\sin\frac{1}{x-1} \leq |x-1|$)

$-|x-1| \to 0$ as $x\to 1^-$; $(x-1)\sin\frac{1}{x-1} \to 0$ as $x\to 1^-$ by ST; $|x-1| \to 0$ as $x\to 1^-$

And $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax+2) = a+2$

Since $f(x)$ is cont, we have $\lim_{x\to 1^+} f(x) = a+2 = 2 = \lim_{x\to 1^-} f(x)$. Thus $\boxed{a=0}$

5.(10 pts) Show that $e \ln x \le x$ for all $x > 0$.

Define $f(x) = x - e\ln x$ on $(0,\infty)$.

Then $f'(x) = \frac{x-e}{x}$.

Therefore, $f'(x) < 0$ for all $x \in (0,e) \Rightarrow f$ is decreasing
$f'(x) > 0$ for all $x \in (e,\infty) \Rightarrow f$ is increasing.

Then, if $x \in (0,e) \Rightarrow f(x) > f(e) \Rightarrow x - e\ln x > 0 \Rightarrow x > e\ln x$
if $x \in (e,\infty) \Rightarrow f(x) > f(e) \Rightarrow x - e\ln x > 0 \Rightarrow x > e\ln x$

And at $x=e$, we have the equality. Thus,

$e\ln x \le x \quad \forall x > 0$

6.(12pts) A particle is moving around the ellipse $4x^2 + 16y^2 = 64$. At any time t, its x- and y-coordinates are given by $x(t) = 4\cos t$ and $y(t) = 2\sin t$. At what rate is the particle's distance to the point $(2,0)$ changing at any time t? At what rate is the distance changing when $t = \frac{\pi}{2}$?

$s(t)$: distance from particle to point $(2,0)$ at any time t

$s^2 = (x-2)^2 + (y-0)^2$, by differentiating wrt t we have

$2s\frac{ds}{dt} = 2(x-2)\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{ds}{dt} = \frac{x-2}{s}\frac{dx}{dt} + \frac{y}{s}\frac{dy}{dt}$, and since $x(t) = 4\cos t \Rightarrow \frac{dx}{dt} = -4\sin t$
$y(t) = 2\sin t \Rightarrow \frac{dy}{dt} = 2\cos t$

we have $\frac{ds}{dt} = \frac{4\cos t - 2}{s}(-4\sin t) + \frac{2\sin 2t}{s} = \frac{1}{s}(8\sin t - 6\sin 2t)$

rate of change in distance between the particle and the point $(2,0)$ at t.

At $t = \frac{\pi}{2} \Rightarrow s^2\Big|_{t=\frac{\pi}{2}} = \left[(x-2)^2 + y^2\right]\Big|_{t=\frac{\pi}{2}} = \left[(4\cos t - 2)^2 + (2\sin t)^2\right]\Big|_{t=\frac{\pi}{2}} = 8 \Leftrightarrow \boxed{s = 2\sqrt{2}}$

Then $\frac{ds}{dt}\Big|_{t=\frac{\pi}{2}} = \frac{1}{2\sqrt{2}}\left(8\sin\frac{\pi}{2} - 6\sin\pi\right) = 2\sqrt{2}$

$$\boxed{\frac{ds}{dt}\Big|_{t=\frac{\pi}{2}} = 2\sqrt{2}}$$

M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	Mid Term II	02.05.2015	13:30
Last Name : Name : Student No:	Signature : Duration : 100 *minutes*			
6 QUESTIONS ON 4 PAGES			TOTAL 100 POINTS	
1 | 2 | 3 | 4 | 5 | 6	SHOW YOUR WORK			

1. (5+5=10 pts)

(a) Evaluate $\lim_{x\to 1^+}\left(\frac{4}{\pi}\arctan x\right)^{\left(\frac{3}{x^2+2x-3}\right)}$ $= [1^\infty]$

let $y=\left(\frac{4}{\pi}\arctan x\right)^{\frac{3}{x^2+2x-3}}$

$$\ln y = \frac{3\ln\left(\frac{4}{\pi}\arctan x\right)}{x^2+2x-3}$$

$\lim_{x\to1^+}\ln y = \left[\frac{0}{0}\right]$

$\Rightarrow \lim_{x\to1^+}\ln y \overset{L'H.}{=} 3\lim_{x\to1^+}\dfrac{\frac{1}{\frac{4}{\pi}\arctan x}\cdot\frac{\frac{4}{\pi}}{1+x^2}}{2x+2}$

$= \frac{3}{2\pi}$

$$\boxed{\lim_{x\to1^+} y = e^{\frac{3}{2\pi}}}$$

(b) Evaluate $\lim_{x\to\pi/2}\dfrac{\int_x^{\pi/2}\ln(\sin t)\,dt}{\sin x-1}$ $= \left[\frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}}\dfrac{-\ln(\sin x)}{\cos x} = \left[\frac{0}{0}\right]$

$\overset{L'H}{=} \lim_{x\to\frac{\pi}{2}}\dfrac{-\frac{1}{\sin x}\cdot\cos x}{-\sin x} = \boxed{0}$

2. (12 pts)

Find all local extreme points of the function $f(x)=\int_0^{\frac{1}{3}x^3-\frac{3}{2}x^2-4x+7}\sin\left(\frac{1}{1+t^4}\right)\,dt$

By F.T.C. Part I $f'(x) = \sin\left(\dfrac{1}{1+\left(\frac{1}{3}x^3-\frac{3}{2}x^2-4x+7\right)^4}\right)(x^2-3x-4)$

$f'(x)=0 \iff x=4$ or $x=-1$

		-1		4	
$f'(x)$	$+$	0	$-$	0	$+$
	$\nearrow$		$\searrow$		$\nearrow$

at $x=-1$ f has a local maximum.

at $x=4$ f has a local minimum.

$u = \sec x$ $\quad$ $v = \tan x$

$du = \tan x \sec x\, dx$ $\quad$ $dv = \sec^2 x\, dx$

3.(5+5+5+5+7=27 pts)

(a) Find $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x\sqrt{\sec x} + \sec x\sqrt{\tan x}}{\cos x}\, dx$ $\displaystyle = \int_0^{\pi/3} \tan x \cdot \sec x \cdot \sqrt{\sec x}\, dx + \int_0^{\pi/3} \sec^2 x \sqrt{\tan x}\, dx$

$$= \int_1^2 \sqrt{u}\, du + \int_0^{\sqrt{3}} \sqrt{v}\, dv = \frac{2}{3}\left(u^{3/2}\Big|_1^2 \right) + \frac{2}{3}\left(v^{3/2}\Big|_0^{\sqrt{3}} \right)$$

$$= \boxed{\frac{2^{5/2}}{3} - \frac{2}{3} + 2 \cdot 3^{-1/4}}$$

(b) Find $\displaystyle\int_{-\pi}^{\pi} \left(1 + \sin^3 x\, \sqrt{9x^2 + 4\cos^2 x}\right) dx$ Note that $f(x) = \sin^3 x \sqrt{9x^2 + 4\cos^2 x}$ is ~~odd~~ an odd function.

$$= \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} f(x)\, dx = 2\pi + 0 = \boxed{2\pi}$$

(c) Evaluate $\displaystyle\lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \cos^2\left(\frac{i\pi}{n}\right)$ $\quad$ $\displaystyle\sum_{i=1}^{n} \cos^2\left(\frac{i\pi}{n}\right)\frac{\pi}{n}$ is the Riemann Sum corresponding to the function $f(x) = \cos^2 x$ on $[0, \pi]$ with partition $x_i = \frac{i\pi}{n}$ $i = 0, \ldots, n$

$$= \int_0^{\pi} \cos^2 x\, dx = \frac{1}{2} \int_0^{\pi} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x \Big|_0^{\pi} \right] = \boxed{\frac{\pi}{2}}$$

(d) Evaluate $\displaystyle\int e^{6x} \sin(e^{3x})\, dx$

let $u = e^{3x}$ $\quad$ $dV = e^{3x} \sin e^{3x}\, dx$

$du = 3e^{3x}\, dx$ $\quad$ $V = -\frac{1}{3}\cos(e^{3x})$

$$= -\frac{1}{3} e^{3x} \cos e^{3x} + \int e^{3x} \cos e^{3x}\, dx$$

$$= \boxed{-\frac{1}{3} e^{3x} \cos e^{3x} + \frac{1}{3} \sin e^{3x} + C}$$

(e) Evaluate $\displaystyle\int \frac{x^3 e^{x^2}}{(x^2+1)^2}\, dx$

$u = x^2 e^{x^2}$ $\quad$ $dV = \frac{x}{(x^2+1)^2}\, dx$

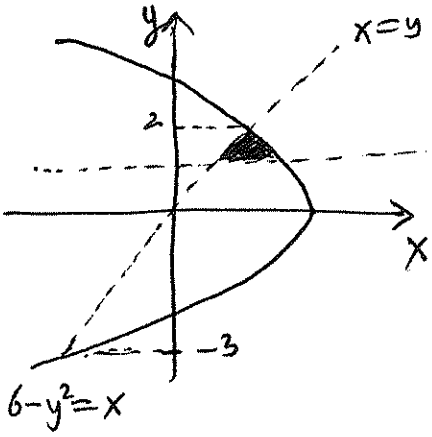
$du = 2x e^{x^2}(1 + x^2)\, dx$ $\quad$ $V = -\frac{1}{2}\left(\frac{1}{x^2+1}\right)$

$$= \frac{-x^2 e^{x^2}}{2(x^2+1)} + \int x e^{x^2}\, dx$$

$$= \boxed{\frac{-x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2} e^{x^2} + C}$$

4.(10+10=20 pts)

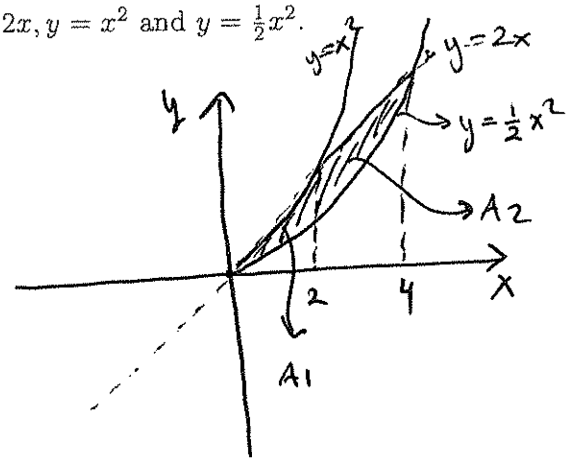
(a) Find the area of the region bounded by $6 - y^2 = x, x = y$ and above $y = 1$.



To find intersections: $6-y^2=y \Rightarrow y=-3$ or $y=2$

$$\text{Area} = \int_1^2 (6-y^2)-y\,dy = \boxed{\frac{13}{6}}$$

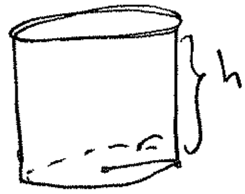
(b) Write (But do **NOT** evaluate) integral(s) which gives the area of the region bounded by the curves $y = 2x, y = x^2$ and $y = \frac{1}{2}x^2$.



$$\text{Area} = \underbrace{\int_0^2 \left(x^2 - \frac{1}{2}x^2\right)dx}_{A_1} + \underbrace{\int_2^4 \left(2x - \frac{1}{2}x^2\right)dx}_{A_2}$$

5.(10 pts) A cylindrical can with volume $20\pi\ m^3$ will be produced. The material for the top and bottom costs 5 TL per square meter and the rest costs 4 TL per square meter. Find the height of the can with minimal cost of material. Justify your answer.

$$\text{Volume} = 20\pi = \pi r^2 h \Rightarrow \boxed{h = \frac{20}{r^2}}$$



$$\text{Cost} = (\text{Cost of top+bottom}) + (\text{Cost of sides})$$

$$= 5(2\pi r^2) + 4(2\pi r h)$$

$$C(r) = 10\pi r^2 + 8\pi r \cdot \frac{20}{r^2} = 10\pi r^2 + \frac{160\pi}{r} \quad (r>0)$$

$$C'(r) = 20\pi r - \frac{160\pi}{r^2} = 0 \Leftrightarrow \boxed{r=2}$$

	2	
$C'(r)$	$-$	$+$
	$\searrow$	$\nearrow$

so $\boxed{h=5}$

so at $r=2$ $C(r)$ has abs. min.

6.(3+3+5+5+5=21 pts) Consider the function $f(x) = \dfrac{x^2-2}{(x-1)^2}$.

(a) Find the domain and intercepts of f.

Domain of $f = \mathbb{R} - \{1\}$

$x=0 \Rightarrow y = \cancel{2}\,2$ $\quad (0,-2)$ y-intercept

$y=0 \Rightarrow x = \mp\sqrt{2}$ $\quad (\mp\sqrt{2}, 0)$ x-intercepts

(b) Find all asymptotes of f.

$\lim\limits_{x\to\mp\infty} f(x) = 1 \Rightarrow y=1$ is a horizontal asymptote

$\lim\limits_{x\to 1^{\mp}} f(x) = -\infty \Rightarrow x=1$ is a vertical asymptote.

(c) Find the intervals where f is increasing/decreasing and all local extreme points of f.

$$f'(x) = \frac{4-2x}{(x-1)^3}$$

		1		2	
$f'(x)$	$-$		$+$	0	$-$
	$\searrow$		$\nearrow$		$\searrow$

at $x=2$ f has a local maximum.

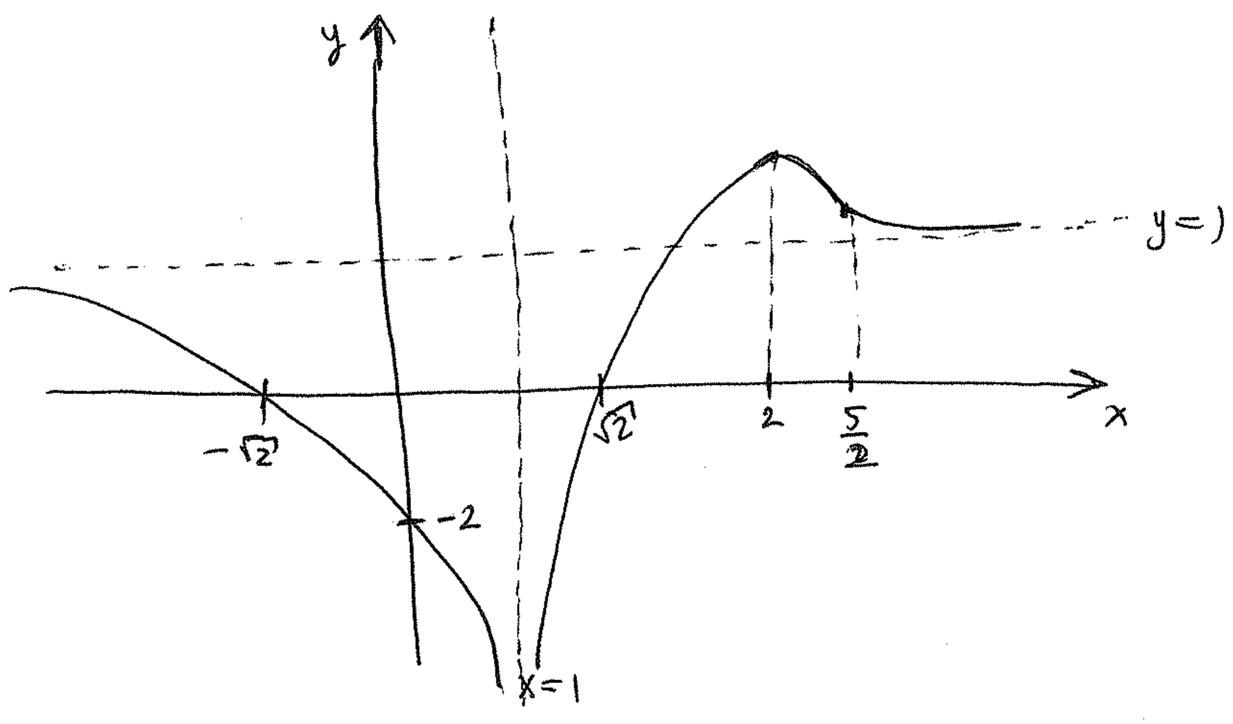
(d) Find the intervals where f is concave up/down and inflection points of f.

$$f''(x) = \frac{4x-10}{(x-1)^4}$$

		1		5/2	
$f''(x)$	$-$		$-$	0	$+$
	conc. down.		conc. down.		conc. up.

at $x = 5/2$ f has an inflection point.

(e) Using the information you obtained in the previous parts, sketch the graph of f. Indicate clearly the intercepts, asymptotes, local extreme values and inflection points of f in your sketch.



M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	MidTerm I	08.04.2017	13:30
Last Name : Name : Student No:	Signature : Section : Duration : 100 *minutes*			
6 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS		

1	2	3	4	5	6	SHOW YOUR WORK		

1. (7+8+10 pts) This question has unrelated parts.

(a) Evaluate $\frac{d}{dx}\left[\arctan\left(\frac{x^2+e^x}{\ln(x^2+5)}\right)\right]$ (Do not simplify your answer).

$$\frac{d}{dx}\left[\arctan\left(\frac{x^2+e^x}{\ln(x^2+5)}\right)\right]=\frac{1}{1+\left(\frac{x^2+e^x}{\ln(x^2+5)}\right)^2}\cdot\frac{(2x+e^x)\cdot\ln(x^2+5)-(x^2+e^x)\cdot\frac{2x}{x^2+5}}{(\ln(x^2+5))^2}$$

(b) Find the equation of the tangent line to $f^{-1}(x)$ at $x=2\pi$ if $f(x)=2x-\sin(x)$.

$$(f^{-1})'(2\pi)=\frac{1}{f'(f^{-1}(2\pi))}=\frac{1}{2-\cos(\pi)}=\frac{1}{2-(-1)}=\frac{1}{3}$$

$$\therefore\ y-\pi=\frac{1}{3}(x-2\pi)\Rightarrow y=\frac{1}{3}x+\frac{\pi}{3}$$

$f'(x)=2-\cos x>0$, hence f is one to one and $f^{-1}(x)$ exists

$f^{-1}(2\pi)=a \Leftrightarrow f(a)=2\pi$, hence $a=\pi$

(c) Suppose that y is a function of x near the point $(1,-1)$. Find $\frac{dy}{dx}$ at the point $(1,-1)$ if

$$1+xy^2=\sec(x+y)+x^{y+1}.$$

We'll apply implicit derivative, but first we can write $x^{y+1}=e^{\ln x\cdot(y+1)}$

$$\frac{d}{dx}(1+xy^2)=\frac{d}{dx}\left(\sec(x+y)+e^{\ln x\cdot(y+1)}\right)$$

$$y^2+2xy\frac{dy}{dx}=\sec(x+y)\cdot\tan(x+y)\left(1+\frac{dy}{dx}\right)+e^{\ln x\cdot(y+1)}\cdot\left(\frac{y+1}{x}+\ln x\cdot\frac{dy}{dx}\right)$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)}=\frac{\sec(x+y)\cdot\tan(x+y)+e^{\ln x\cdot(y+1)}\cdot\left(\frac{y+1}{x}\right)-y^2}{2xy-\sec(x+y)\cdot\tan(x+y)-e^{\ln x\cdot(y+1)}\cdot\ln x}$$

$$=\frac{\sec(0)\cdot\tan(0)+e^0\cdot 0-(-1)^2}{2\cdot1\cdot(-1)-\sec(0)\cdot\tan(0)-e^0\cdot 0}=\frac{1}{2}.$$

Question 2. (6+6+6+7 pts) Evaluate the following limits. (DO NOT use L'Hospital's)

(a) $\lim_{x\to\infty}(\sqrt{x+\sqrt{x}}-\sqrt{x}) = \lim_{x\to\infty} \dfrac{(\sqrt{x+\sqrt{x}}-\sqrt{x})(\sqrt{x+\sqrt{x}}+\sqrt{x})}{(\sqrt{x+\sqrt{x}}+\sqrt{x})}$

$= \lim_{x\to\infty} \dfrac{\cancel{x}+\sqrt{x}-\cancel{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}} = \lim_{x\to\infty} \dfrac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}$

$= \lim_{x\to\infty} \dfrac{1}{\sqrt{1+\cancel{\frac{1}{x^{1/2}}}}+1} = \dfrac{1}{2}$ as $\dfrac{1}{x^{1/2}} \to 0$ when $x\to\infty$

(b) $\lim_{x\to\infty} \dfrac{e^x}{\sinh(x)} = \lim_{x\to\infty} \dfrac{e^x}{\frac{e^x-e^{-x}}{2}} = \lim_{x\to\infty} \dfrac{2e^x}{\cancel{e^x}(1-e^{-2x})}$

$= \lim_{x\to\infty} \dfrac{2}{1-e^{-2x}} = 2$ as $e^{-2x} \to 0$ when $x\to\infty$

(c) $\lim_{x\to 1} \dfrac{2x-2}{|x^3-x^2|}$ $\left(\dfrac{0}{0}\right)$

$$|x^3-x^2| = \begin{cases} -(x^3-x^2) & x<0 \\ -(x^3-x^2) & 0\leq x<1 \\ x^3-x^2 & 1\leq x \end{cases}$$

$\lim_{x\to 1^+} \dfrac{2x-2}{|x^3-x^2|} = \lim_{x\to 1^+} \dfrac{2\cancel{(x-1)}}{x^2\cancel{(x-1)}} = \dfrac{2}{1^2} = 2$

$2/x^2$ is cont. at 1

$\lim_{x\to 1^-} \dfrac{2x-2}{|x^3-x^2|} = \lim_{x\to 1^-} \dfrac{2\cancel{(x-1)}}{-x^2\cancel{(x-1)}} = \dfrac{2}{-1^2} = -2$

$2/-x^2$ is cont. at 1

$\Rightarrow \lim_{x\to 1} \dfrac{2x-2}{|x^3-x^2|}$ doesn't exist.

(d) $\lim_{x\to 0} \dfrac{2^x-1}{\sin 4x}$

$\lim_{x\to 0} \dfrac{2^x-1}{\sin 4x} = \lim_{x\to 0} \dfrac{2^x-1}{x}\cdot\dfrac{x}{\sin 4x} = \lim_{x\to 0} \dfrac{2^x-2^0}{x-0} \cdot \lim_{x\to 0} \dfrac{(4x)}{4\cdot\sin(4x)}$

since each limit exists

$= (2^0\cdot\ln 2)\cdot\left(\dfrac{1}{4}\cdot 1\right) = \dfrac{\ln 2}{4}$

since $\lim_{x\to 0} \dfrac{2^x-2^0}{x-0} = \dfrac{d}{dx}(2^x)\Big|_{x=0} = (2^x\cdot\ln 2)\Big|_{x=0} = 2^0\cdot\ln 2 = \ln 2$

$\lim_{x\to 0} \dfrac{1}{4}\dfrac{(4x)}{\sin(4x)} = \dfrac{1}{4}\lim_{u\to 0} \dfrac{u}{\sin(u)} = \dfrac{1}{4}$

$u=4x$

$x\to 0,\ u\to 0$

3.(3+5 pts) (a) Write the formal definition (the ϵ-δ definition) of $\lim_{x\to a} f(x) = L$.

For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

(b) Show that $\lim_{x\to 119}(2x+1) = 239$ by using the formal definition of the limit.

For every $\varepsilon > 0$, choose $\delta = \varepsilon/2$. Then, if $0 < |x-119| < \delta = \frac{\varepsilon}{2}$

$$|2x+1-239| = |2x-238| = 2|x-119| < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$$

4.(8+9 pts) Let $f(x) = \begin{cases} 2xe^{x^2} & \text{if } x \le 0 \\ x^{3/2}\cos\left(\frac{1}{\sqrt{x}}\right) + 2x & \text{if } x > 0 \end{cases}$

(a) Show that $f(x)$ is continuous everywhere.

If $x \neq 0$, then $2xe^{x^2}$ & $x^{3/2}\cdot\cos\left(\frac{1}{\sqrt{x}}\right) + 2x$ are continuous since $2xe^{x^2}$'s domain contains $(-\infty, 0)$ and $\left(x^{3/2}\cos(1/\sqrt{x}) + 2x\right)$'s domain contains $(0, +\infty)$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2xe^{x^2} = 2\cdot 0\cdot e^{0^2} = 0 = f(0)$, since $2xe^{x^2}$ is continuous on $(-\infty, \infty)$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{3/2}\cos\left(\frac{1}{\sqrt{x}}\right) + \lim_{x\to 0^+} 2x = 0 + 2\cdot 0 = 0 = f(0)$ $\therefore$ Cont. at $x=0$, too.

since $2x$ is continuous at 0 and $-x^{3/2} \le x^{3/2}\cos\left(\frac{1}{\sqrt{x}}\right) \le x^{3/2}$ since $-1 \le \cos(1/\sqrt{x}) \le 1$

when $x \to 0^+$ by squeeze

0

(b) Is $f(x)$ differentiable at $x = 0$? Explain your answer.

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^{3/2}\cdot\cos(1/\sqrt{x}) + 2x - 0}{x - 0}$$

$$= \lim_{x\to 0^+} x^{1/2}\cdot\cos\left(\frac{1}{\sqrt{x}}\right) + 2 = 2$$

since $1 \le \cos(1/\sqrt{x}) \le 1$, and $-x^{1/2} + 2 \le x^{1/2}\cos(1/\sqrt{x}) + 2 \le x^{1/2} + 2$

when $x \to 0^+$ since $\mp x^{1/2} + 2$ is cont. by squeeze thm

0

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{2xe^{x^2} - 0}{x - 0} = \lim_{x\to 0^-} 2e^{x^2} = 2$$

$2e^{x^2}$ is cont. at 0.

$\therefore f'(0) = 2$, $f(x)$ is diff'ble at $x = 0$.

5.(12 pts) Consider the function $f(x) = x^3 - 27x + a$ where $a \in \mathbb{R}$ is a constant. Show that f does **NOT** have two zeros in the interval $[0, 1]$.

$f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ since it's a polynomial.

If there were two zeros of $f(x)$ on $[0,1]$ say $f(a)=0$, $f(b)=0$, $a,b \in [0,1]$

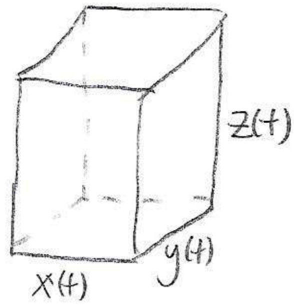
then, by Rolle's (or Mean Value) Theorem, there exists $c \in (0,1)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0$. However, $f'(x) = 3x^2 - 27$

has zeros at $x = 3$ and $x = -3$ but $3, -3 \notin [0,1]$.

Hence, $f(x)$ can't have two zeros in $[0,1]$.

6.(13 pts) The height of a rectangular box decreases at $1cm$ per hour, its width decreases at $0.5cm$ per hour and its length decreases at $2cm$ per hour. When the shape of the box becomes a cube, its volume decreases at the rate of $14cm^3$ per hour. Find the volume of the box when it **becomes a cube**.

Volume of a box is given by $V(t) = x(t) \cdot y(t) \cdot z(t)$



$$\frac{dV}{dt} = \frac{dx}{dt} \cdot y(t) \cdot z(t) + x(t) \frac{dy}{dt} \cdot z(t) + x(t) \cdot y(t) \cdot \frac{dz}{dt}$$

When $x(t) = y(t) = z(t) = a$, we have $\frac{dV}{dt} = -14 \, cm^3/h$

and $\frac{dx}{dt} = -1 \, cm/h$, $\frac{dy}{dt} = -0.5 \, cm/h$, $\frac{dz}{dt} = -2 \, cm/h$

Hence, we'll have

$$-14 = -1 \cdot a \cdot a - 0.5 \cdot a \cdot a - 2 \cdot a \cdot a = -3.5a^2$$

$$a^2 = 4 \Rightarrow a = 2 \, cm$$

When it's a cube, we have the volume $V = 2 \cdot 2 \cdot 2 = 8 \, cm^3$.

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M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	MidTerm II	13.05.2017	13:30

Last Name : Name : Student No:	Signature : Section : Duration : 119 *minutes*	

5 QUESTIONS ON 4 PAGES	TOTAL 100 POINTS

1	2	3	4	5	SHOW YOUR WORK		

1. (3+4+6+6+6 pts) Given $f(x) = \dfrac{x^3-1}{x^3+1}$ with $f'(x) = \dfrac{6x^2}{(x^3+1)^2}$ and $f''(x) = \dfrac{12x(1-2x^3)}{(x^3+1)^3}$.

(a) Find the domain and intercepts of $f(x)$.

$\mathrm{Dom}(f) = \mathbb{R}\setminus\{-1\}$, x-intercept: $y=0$, $x=1$

y-intercept: $x=0$, $y=-1$

(b) Find all asymptotes of $f(x)$.

Vertical Asymptote: $\lim\limits_{x\to -1^+} \dfrac{x^3-1}{x^3+1} = -\infty$ $\quad \lim\limits_{x\to -1^-} \dfrac{x^3-1}{x^3+1} = +\infty$ i.e. $x=-1$

Horizontal Asymptote: $\lim\limits_{x\to \mp\infty} \dfrac{x^3-1}{x^3+1} = \lim\limits_{x\to \mp\infty} \dfrac{\cancel{x^3}(1-1/x^3)}{\cancel{x^3}(1+1/x^3)} = 1$ i.e. $y=1$

(c) Find the intervals of increase and decrease and the local extreme values of $f(x)$.

$f'(x)=0 \Rightarrow x=0$, $f'(x)$ doesn't exist when $x=-1$

		-1		0	
$f'(x)$	$+$ ↗		$+$ ↗		$+$ ↗

Always increasing on $(-\infty,-1)\cup(-1,+\infty)$

No local extremas.

(d) Find the intervals of concavity and the inflection points of $f(x)$.

$f''(x)=0 \Rightarrow x=0$, $x=\dfrac{1}{\sqrt[3]{2}}$, $f''(x)$ doesn't exist when $x=-1$

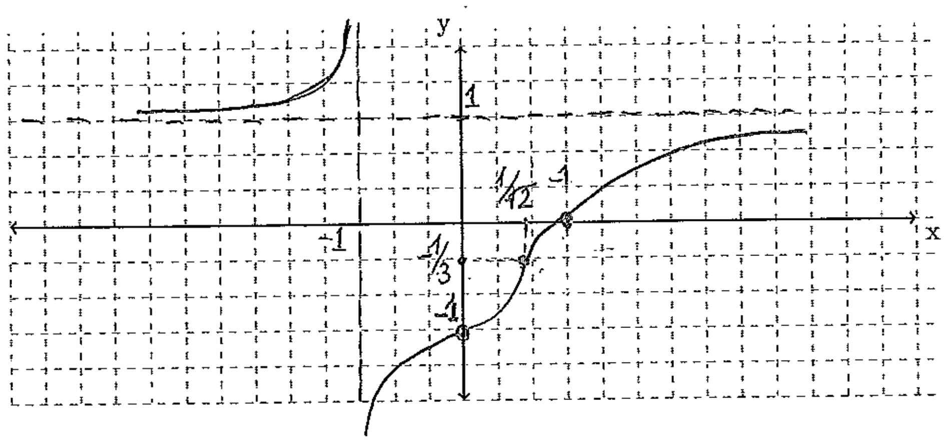
		-1		0		$1/\sqrt[3]{2}$	
f''	$+$		$-$		$+$		$-$

Concave up on $(-\infty,-1)\cup(0, 1/\sqrt[3]{2})$

Concave down on $(-1,0)\cup(1/\sqrt[3]{2}, +\infty)$

0 & $1/\sqrt[3]{2}$ are inflection pts.

(e) Sketch the graph of $f(x)$.



$f(0) = -1$

$f(1/\sqrt[3]{2}) = -1/3$

Question 2. (6+6+7+6 pts) Evaluate the following limits.

(a) $\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{\ln x\cdot(x-1)} \left(\frac{0}{0}\right) \overset{L'Hos.}{=} \lim_{x\to1^+} \frac{1-1/x}{\frac{1}{x}(x-1)+\ln x} \left(\frac{0}{0}\right)$

$\overset{L'Hos.}{=} \lim_{x\to1^+} \frac{1/x^2}{+1/x^2+\frac{1}{x}} \underset{\text{cont. at } 1}{=} \frac{1}{2}.$

(b) $\lim_{x\to0^+}(1-3x)^{\frac{1}{x}}\ (1^\infty) = \lim_{x\to0^+}\left(e^{\ln(1-3x)}\right)^{1/x} = \lim_{x\to0^+} e^{\frac{\ln(1-3x)}{x}} = e^{-3}.$

e^x is cont. at -3

$\left[\lim_{x\to0^+}\frac{\ln(1-3x)}{x}\left(\frac{0}{0}\right) = \lim_{x\to0^+}\frac{\frac{-3}{1-3x}}{1} \underset{\text{cont. at } 0}{=} -3\right]$

(c) $\lim_{x\to0}\frac{\sqrt{1-\tan x}-\sqrt{1+\tan x}}{\sin x}\ \frac{(\sqrt{1-\tan x}+\sqrt{1+\tan x})}{(\sqrt{1-\tan x}+\sqrt{1+\tan x})} = \lim_{x\to0}\frac{(1-\tan x)-(1+\tan x)}{\sin x(\sqrt{1-\tan x}+\sqrt{1+\tan x})}$

$= \lim_{x\to0}\frac{-2\cancel{\tan x}}{\cancel{\sin x}\cdot(\sqrt{1-\tan x}+\sqrt{1+\tan x})} = \lim_{x\to0}\frac{-2}{\cos x\cdot(\sqrt{1-\tan x}+\sqrt{1+\tan x})}$

$\underset{\text{cont. at } 0}{=} -1$

(d) $\lim_{x\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{2+\frac{3i}{n}} = \int_a^b f(x)\,dx$, where

$\Delta x = \frac{3}{n} = \frac{b-a}{n} \qquad x_i = a+i\cdot\Delta x = a+\frac{3i}{n}$

$f(x_i)=\sqrt{2+\frac{3i}{n}} = \sqrt{2+i\Delta x} = \sqrt{x_i} \Rightarrow f(x)=\sqrt{x},\ a=2,\ b=5$

$\int_2^5\sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_2^5 = \frac{2}{3}\left(\sqrt{125}-\sqrt{8}\right)$

Hence, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{2+\frac{3i}{n}} = \frac{2}{3}\left(\sqrt{125}-\sqrt{8}\right)$

3.(6+6+6+7 pts) Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^2} = \int \frac{2(u-1)\,du}{u^2} = 2\int\left(\frac{1}{u}-\frac{1}{u^2}\right)du = 2\left(\ln|u| + \frac{1}{u}\right) + C$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2(u-1)\,du$

$$= 2\left(\ln(1+\sqrt{x}) + \frac{1}{1+\sqrt{x}}\right) + C$$

(b) $\displaystyle\int_0^{\ln 2} e^{2x}e^{(e^x)}\,dx = \int_1^2 t e^t\,dt = e^t\cdot t\Big|_1^2 - \int_1^2 e^t\,dt = (e^t\cdot t - e^t)\Big|_1^2$

$t = e^x \quad dt = e^x\,dx$

$u = t \quad du = dt$

$dv = e^t\,dt \quad v = e^t$

$$= (2e^2 - e^2) - (e - e) = e^2.$$

(c) $\displaystyle\int \frac{\cos x + \cos^3 x}{\sin^3 x - \sin^2 x}\,dx = \int \frac{du}{u^3-u^2} + \int \frac{(1-u^2)\,du}{u^3-u^2} = \int \frac{-u^2+2}{u^2(u-1)}\,du$

$u = \sin x$

$du = \cos x\,dx$

$\displaystyle\frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} = \frac{-u^2+2}{u^2(u-1)} \Rightarrow \frac{Au^2 + Cu^2 - Au + Bu - B}{u^2(u-1)} = \frac{-u^2+2}{u^2(u-1)} \Rightarrow \begin{cases} -B = 2 \\ -A+B = 0 \\ A+C = -1 \end{cases}$

$u(u-1) \quad (u-1) \quad (u^2)$

we get $\left.\begin{array}{l} B=-2 \\ A=-2 \\ C=1 \end{array}\right\} \Rightarrow \displaystyle\int \frac{-u^2+2}{u^2(u-1)}\,du = -2\ln|u| + 2\,\frac{1}{u} + \ln|u-1| + C = \ln\left(\frac{1}{(\sin x)^2}\right) + \frac{2}{\sin x} + \ln|\sin x - 1| + C$

(d) $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \frac{\ln(\sec\theta)\cdot\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta - 1}} = \int \frac{\ln(\sec\theta)\cdot\sec^2\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}}$

$x = \sec\theta$

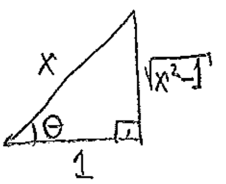
$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \ln(\sec\theta)\cdot\sec^2\theta\,d\theta = \ln(\sec\theta)\cdot\tan\theta - \int\tan^2\theta\,d\theta$$

$u = \ln(\sec\theta) \quad du = \tan\theta\,d\theta$

$dv = \sec^2\theta \quad v = \tan\theta$

$$= \ln|\sec\theta|\cdot\tan\theta - \int(\sec^2\theta - 1)\,d\theta = \ln(\sec\theta)\cdot\tan\theta - \tan\theta + \theta + C$$



$$= \ln(x)\cdot\sqrt{x^2-1} - \sqrt{x^2-1} + \sec^{-1}(x) + C$$

$\theta = \sec^{-1}(x)$

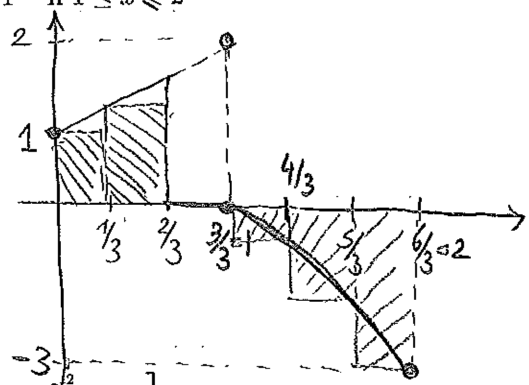
4.(6+9 pts) This problem has two **unrelated** parts.

(a) Compute the lower Riemann sum for the function $f(x) = \begin{cases} x+1 & \text{if } 0 \le x < 1 \\ -x^2+1 & \text{if } 1 \le x \leqslant 2 \end{cases}$ on [0,2] corresponding to partition P_6 of [0,2] into 6 subintervals of equal length.

$$L(f,P_6) = \frac{1}{3}\cdot\left[f(0) + f(1/3) + f(1) + f(4/3) + f(5/3) + f(2)\right]$$

$$= \frac{1}{3}\left[1 + \frac{4}{3} + 0 - \frac{7}{9} - \frac{16}{9} - 3\right]$$

$$= \frac{1}{3}\left[\frac{9+12+0-7-16-27}{9}\right] = \frac{-29}{27}$$



(b) Show that $F(x)$ has a local maximum at $x = -2$, where $F(x) = \int_0^x \left[t^2 \cdot \int_4^{t^2} e^{u^2}\,du\right] dt$.

$F'(x) = x^2 \cdot \int_4^{x^2} e^{u^2}du \quad F'(-2) = (-2)^2 \int_4^{(-2)^2=4} e^{u^2}du = 0 \quad x=-2$ is a critical pt.

$F''(x) = 2x\int_4^{x^2} e^{u^2}du + x^2 \cdot e^{x^4}\cdot 2x \quad F''(-2) = 2\cdot(-2)\int_4^{(-2)^2} e^{u^2}du + (-2)^2 e^{(-2)^4}\cdot(2\cdot(-2))$

$= -16e^{16} < 0$

Hence, by 2^{nd} Derivative Test, $x=-2$ is a local maximum pt.

5.(10 pts) There are 50 orange trees in a garden. Each tree produces 800 oranges. For each additional tree planted in the garden, the output per tree **drops** by 10 oranges. How many trees should be added to the existing garden in order to maximize the total output of trees?

$x \equiv$ number of orange trees, output per tree $= 800 - 10x$

$T(x) = (50+x)(800-10x) = 40000 + 300x - 10x^2 \qquad 0 \le x \le 80$

$T'(x) = 300 - 20x = 0 \Rightarrow x = 15$

$T(x)$ is continuous on $[0,80]$ By Exterome Value Theorem, the absolute max/min exist.

$T(0) = 40000$

$T(80) = 0$

$T(15) = 42250 \rightsquigarrow$ the global maximum value Hence, 15 more trees should be planted to maximize the total.

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M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	

Code : *Math 119* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *November 15.2014* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature :
	7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (6+6+6=18 points)

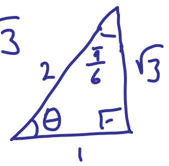
a) Evaluate $\lim\limits_{x\to-\infty} \arctan\left(\dfrac{\sqrt{3}x^2+5x+2}{x^2+1}\right)$. Show your work.
(Note that $\arctan x$ is another way of writing $\tan^{-1} x$.)

Let $y(x) = \dfrac{\sqrt{3}\,x^2+5x+2}{x^2+1}$. Then $\lim\limits_{x\to-\infty} y(x) = \lim\limits_{x\to-\infty} \dfrac{\cancel{x^2}\left(\sqrt{3}+\frac{5}{x}+\frac{2}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)} = \sqrt{3}$

$\frac{5}{x} \to 0$, $\frac{2}{x^2} \to 0$, $\frac{1}{x^2} \to 0$

$\lim\limits_{x\to-\infty} \arctan y(x) = \lim\limits_{y\to\sqrt{3}} \arctan y = \arctan\sqrt{3} = \theta = \pi/3$

$\tan\theta = \sqrt{3}$



b) Without using L'Hopital's Rule, evaluate $\lim\limits_{x\to0^+} \dfrac{(\tan\sqrt{x})\sqrt{2x+1}}{x}$. Show your work.

$\lim\limits_{x\to0^+} \dfrac{(\tan\sqrt{x})\sqrt{2x+1}}{x} = \lim\limits_{x\to0^+} \left(\dfrac{\sin\sqrt{x}}{\cos\sqrt{x}}\ \dfrac{1}{x}\ \sqrt{2x+1}\right) = \lim\limits_{x\to0^+} \left(\dfrac{\sin\sqrt{x}}{\sqrt{x}}\ \dfrac{1}{\cos\sqrt{x}}\ \sqrt{\dfrac{2x+1}{x}}\right)$

$\dfrac{\sin\sqrt{x}}{\sqrt{x}} \to 1$, $\dfrac{1}{\cos\sqrt{x}} \to 1$, $\sqrt{\dfrac{2x+1}{x}} \to \infty$

$= \infty$

c) Evaluate $\lim\limits_{x\to0} \dfrac{\ln(\cos x)}{x^2}$. Show your work.

$\lim\limits_{x\to0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to0} \dfrac{\ln\cos x}{x^2}$ $\left[\frac{0}{0}\right]$ type indeterminacy.

Consider $\lim\limits_{x\to0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to0} \dfrac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim\limits_{x\to0} \dfrac{\sin x}{x} \cdot \dfrac{-1}{2\cos x}$

$= -\lim\limits_{x\to0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to0} \dfrac{1}{2\cos x} = -1 \cdot \dfrac{1}{2} = \dfrac{-1}{2}$

Thus by L'Hop's Rule $\lim\limits_{x\to0} \dfrac{f(x)}{g(x)} = -1/2$

Question 2 (7+7=14 points)

a) Find $f'(x)$ if $f(x) = \tan\left(\dfrac{\sqrt{x}+1}{x^3+x}\right)$. Show your work. Do not simplify after all derivatives are evaluated.

$$f'(x) = \sec^2\left(\frac{\sqrt{x}+1}{x^3+x}\right) \cdot \frac{d}{dx}\left(\frac{\sqrt{x}+1}{x^3+x}\right)$$

$$= \sec^2\left(\frac{\sqrt{x}+1}{x^3+x}\right) \frac{\frac{1}{2\sqrt{x}}(x^3+x) - (\sqrt{x}+1)(3x^2+1)}{(x^3+x)^2}$$

b) Find $f'(0)$ if $f(x) = \begin{cases} \dfrac{x^3 \sin\frac{1}{x}}{\sin x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Show your work.

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h}}{\sin h} = \underbrace{\lim_{h\to 0} \frac{h}{\sin h}}_{= 1} \cdot \underbrace{\lim_{h\to 0} h \cdot \sin\frac{1}{h}}_{= 0}$$

$$= 0$$

$\sin\frac{1}{h}$ is bounded, $\lim_{h\to 0} h = 0$

Hence $f'(0) = 0$

Question 3 (8+8=16 points) Find $f'(x)$ for the following functions. In each part show your work. Do not simplify after all derivatives are evaluated.

a) $f(x) = \dfrac{\ln(1+x^2)}{\sqrt{e^{2x}+5^x}}$

$$f'(x) = \frac{\frac{2x}{1+x^2}\sqrt{e^{2x}+5^x} - \ln(1+x^2)\left[\frac{1}{2\sqrt{e^{2x}+5^x}} \cdot (2e^{2x} + 5^x \cdot \ln 5)\right]}{e^{2x}+5^x}$$

b) $f(x) = (\sin x)^{\cos(x^3)}$

Let $y = (\sin x)^{\cos(x^3)}$. Then

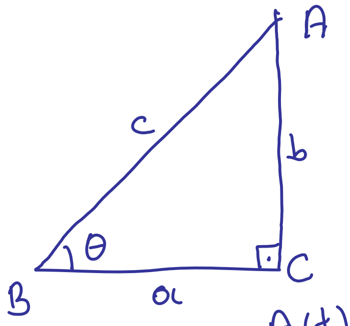
$\ln y = \cos(x^3) \ln(\sin x)$. Take derivative of both sides to get:

$$\frac{1}{y} \cdot y' = -\sin(x^3) \cdot (3x^2) \cdot \ln(\sin x) + \cos(x^3) \frac{1}{\sin x} \cdot \cos x$$

Hence

$$y' = (\sin x)^{\cos(x^3)} \cdot \left[-\sin(x^3) \cdot (3x^2) \cdot \ln(\sin x) + \cos(x^3) \frac{1}{\sin x} \cdot \cos x\right]$$

Question 4 (14 points) Consider the triangle ABC given in the figure. It's sidelenghts a, b, c are changing in such a way that it's area is always $16\,\text{cm}^2$. If the sidelength a is increasing at a rate of $\frac{1}{2}$ cm/sec when a is 8 cm, at what rate (radian/sec) is the measure θ of the angle $\widehat{B}$ changing at that moment?



Let t_0 be the moment $a(t_0) = 8$ cm

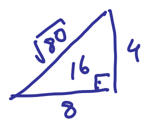
$$\left.\begin{aligned} &\text{Area} = A(t) = \frac{a(t)\cdot b(t)}{2} = 16\ \text{cm}^2 \ \forall t \\ &\left.\frac{d\,a(t)}{dt}\right|_{t_0} = +\frac{1}{2}\ \text{cm/sec.} \end{aligned}\right\} \quad \left.\frac{d\theta(t)}{dt}\right|_{a=8} = ?$$

Since $A(t) = 16\ \text{cm}^2\ \forall t$, we have $b(t) = \dfrac{32}{a(t)}$

$$\tan\theta(t) = \frac{b(t)}{a(t)} = \frac{32}{a^2(t)}.$$ Differentiate wrt t to obtain

$$\sec^2\theta(t)\cdot\theta'(t) = -64\,\frac{1}{a^3(t)}\cdot a'(t)\ \forall t.$$

At $t = t_0$, we have $a(t_0) = 8$, $a'(t_0) = \frac{1}{2}$, $b(t_0) = \frac{32}{8} = 4$ and



$\sec\theta(t_0) = \dfrac{\sqrt{80}}{8}$. Thus $\dfrac{80}{64}\cdot\theta'(t_0) = \cancel{-64}\cdot\dfrac{1}{8\cancel{7}}\ \dfrac{1}{2}$

$$\theta'(t_0) = -\frac{1}{16}\cdot\frac{{}^{4}\cancel{64}}{80} = \frac{-1}{20}\ \text{radian/sec.}$$

Question 5 (4+8=12 points) Let f be a continuous function on $[0, 2]$ and assume that f' and f'' exists on $(0, 2)$.

a) If $f(0) = 0$ and $f(1) = 1$ then show that there exists a number $a \in (0, 1)$ such that $f'(a) = 1$.

Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, by Mean Value Theorem, we know that there is some $\alpha \in (0,1)$ such that

$$1 = \frac{f(1) - f(0)}{1 - 0} = f'(\alpha)$$

b) In addition conditions in part (a) if $f(2) = 2$ show that there exists a number $c \in (0, 2)$ such that $f''(c) = 0$.

Since f is continuous on $[1,2]$ and differentiable on $(1,2)$, by MVT, $\exists\, b \in (1,2)$ such that $f'(b) = \dfrac{f(2) - f(1)}{2-1} = \dfrac{2-1}{2-1} = 1$

Note that since $a \in (0,1)$ and $b \in (1,2)$, we have $a \neq b$. Moreover,

Since $f'(\alpha) = 1 = f'(b)$, and

$$\left.\begin{aligned} &f'(x) \text{ is continuous on } [a,b], \text{ and} \\ &f'(x) \text{ is differentiable on } (a,b) \end{aligned}\right\} \Leftarrow \text{as } f'' \text{ exists on } (0,2) \text{ and } (a,b) \subsetneq (0,2)$$

by Rolle's Thm applied to $f'(x)$, $\exists\, c \in (a,b) \subsetneq (0,2)$ such that $f''(c) = 0$

Question 6 (14 points)

Let C be the graph of the equation $3y^2x - x^3 - y^3 - y = 0$. Find all the points on C through which the tangent lines to C are horizontal.

Using implicit differentiation, we get

$6yy'x + 3y^2 - 3x^2 - 3y^2y' - y' = 0 \Rightarrow \boxed{y'(6xy - 3y^2 - 1) = 3(x^2 - y^2)}$ (*)

We are looking for points (x,y) satisfying given equation and at which $y' = 0$

$\Rightarrow x^2 = y^2 \Rightarrow x = \mp y$. From the given equation we obtain:

- $x = y \Rightarrow 3x^3 - x^3 - x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, x_2 = -1, x_3 = 1$

$\Rightarrow$ pts are $(0,0), (-1,-1), (+1,+1)$.

- $x = -y \rightarrow 3x^3 - x^3 + x^3 + x = 0 = x(3x^2 + 1) = 0 \Rightarrow x = 0 = -y$

$\Rightarrow$ pts are $(0,0)$

$\therefore \{(0,0), (-1,-1), (1,1)\}$ is the set of all points on C, at which $y' = 0$.

Note

Note that at none of these 3 points, the expression $(6xy - 3y^2 - 1)$ in (*) above is zero.

Hence at each of " " " ", y' is defined and equal to zero.

Question 7 (5+7=12 points)

Assume that f is a function such that $f'(x) = \dfrac{x-1}{x^3+1}$ on the interval $(0, \infty)$.

a) Calculate $\dfrac{d}{dx}\left(f(\dfrac{1}{x})\right)$ for $x > 0$.

$\dfrac{d}{dx}\left(f\left(\frac{1}{x}\right)\right) = \dfrac{-1}{x^2} f'\left(\frac{1}{x}\right)$ by Chain Rule.

$= \dfrac{-1}{x^2} \dfrac{\frac{1}{x} - 1}{\frac{1}{x^3} + 1} = \dfrac{+1}{\cancel{x^2}} \dfrac{\frac{x-1}{\cancel{x}}}{\frac{x^3+1}{\cancel{x^3}}} = \dfrac{x-1}{x^3+1}$

b) Prove that $f(x) = f(\dfrac{1}{x})$ for all $x > 0$. (DO NOT use integral.)

Since $\dfrac{d}{dx}\left[f(x) - f\left(\frac{1}{x}\right)\right] = 0 \ \forall x \in (0, \infty)$, (by part (a))

we have $f(x) - f\left(\frac{1}{x}\right) = c \ \forall x \in (0, \infty)$

Put $x = 1$: $f(1) - f(1) = 0$, hence $c = 0$

$\therefore f(x) - f\left(\frac{1}{x}\right) = 0$ ie, $f(x) = f\left(\frac{1}{x}\right) \ \forall x > 0$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 2	

Code : *Math 119* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *December 20.2014* Time : *9:30* Duration : *140 minutes*	Last Name : Name : Student No. : Department : Section : Signature :
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (9+9=18 points) Evaluate the following integrals.

a) $\displaystyle\int_{\pi/2}^{\pi} \cos^5 x \sin^2 x \, dx.$ $= \displaystyle\int_{\pi/2}^{\pi} (\cos^2 x)^2 \sin^2 x \cos x \, dx$

$= \displaystyle\int_{\pi/2}^{\pi} (1-\sin^2 x)^2 \sin^2 x \cos x \, dx = \int_1^0 (1-u^2)^2 u^2 \, du$

$u = \sin x$, $du = \cos x\,dx$ $\quad x = \pi/2 \to u = \sin\frac{\pi}{2} = 1$, $\quad x = \pi \to u = \sin \pi = 0$

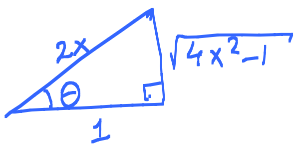
$= \displaystyle\int_1^0 u^2 - 2u^4 + u^6 \, du = \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\Big|_1^0$

$= \underset{(35)}{\dfrac{-1}{3}} + \underset{(21)}{\dfrac{2}{5}} - \underset{(15)}{\dfrac{1}{7}} = \dfrac{-35+42-15}{105} = \dfrac{-8}{105}$

b) $\displaystyle\int \frac{x^2}{(4x^2-1)^{3/2}} \, dx.$ (Assume that $(x > 1/2)$)

$x = \frac{1}{2}\sec\theta$; $\theta \in [0, \pi/2)$

$dx = \frac{1}{2}\sec\theta \tan\theta \, d\theta$



$\tan\theta = \sqrt{4x^2-1}$

$= \displaystyle\int \frac{\frac{1}{4}\sec^2\theta}{\tan^3\theta} \frac{1}{2}\sec\theta\tan\theta \, d\theta = \frac{1}{8}\int \frac{\sec^3\theta}{\tan^2\theta} d\theta = \frac{1}{8}\int \frac{1}{\cancel{\cos^3\theta}} \frac{\cancel{\cos^2\theta}}{\sin^2\theta} d\theta$

$= \displaystyle\frac{1}{8}\int \frac{1}{\cos\theta \sin^2\theta} d\theta = \frac{1}{8}\int \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin^2\theta} d\theta =$

$= \displaystyle\frac{1}{8}\left[\int \frac{1}{\cos\theta} d\theta + \int \frac{\cos\theta}{\sin^2\theta} d\theta\right] = \frac{1}{8}\left[\int \sec\theta \, d\theta + \int \frac{1}{u^2} du\right]$

$u = \sin\theta$, $du = \cos\theta \, d\theta$

$= \displaystyle\frac{1}{8}\left[\ln|\sec\theta + \tan\theta| - \frac{1}{\sin\theta}\right] + C = \frac{1}{8}\left[\ln\left|2x + \sqrt{4x^2-1}\right| - \frac{2x}{\sqrt{4x^2-1}}\right] + C$

Question 2 (9+9=18 points) Evaluate the following integrals.

a) $\int_1^4 \sqrt{x}e^{\sqrt{x}}\,dx. = \int_1^4 \sqrt{x}\, e^{\sqrt{x}}\, 2\sqrt{x}\, \frac{1}{2\sqrt{x}}\,dx = \int_1^2 t\, e^t\, 2t\, dt$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}dx \qquad x=1 \Rightarrow t=1$

$x=4 \Rightarrow t=2$

$= 2\int_1^2 \underbrace{t^2}_{u}\, \underbrace{e^t\,dt}_{dv} = 2\left[\left(t^2 e^t\right)_{t=1}^{t=2} - 2\int_1^2 \underbrace{t}_{u}\, \underbrace{e^t\,dt}_{dv} \right]$

$du = 2t\,dt \qquad du = dt$

$v = e^t \qquad v = e^t$

$= 2\left(4e^2 - e\right) - 4\left[\left(te^t\right)_1^2 - \int_1^2 e^t\,dt \right]$

$= 8e^2 - 2e - 4\left[2e^2 - e - e^t\big|_1^2\right] = \cancel{8e^2} - 2e - \cancel{8e^2} + \cancel{4e} + 4e^2 - \cancel{4e}$

$= 4e^2 - 2e$

b) $\int \frac{x^2+11x+10}{x^3+4x^2+5x}\,dx.$ (Assume that $(x \neq 0)$)

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$= \int \frac{x^2+11x+10}{x\,(x^2+4x+5)}\,dx = \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+4x+5}\right)dx$

$\Delta = 4^2 - 4.5 < 0$

$x^2+11x+10 = A(x^2+4x+5) + (Bx+C)\,x = x^2\underbrace{(A+B)}_{=1} + x\underbrace{(4A+C)}_{=11} + \underbrace{(5A)}_{=10}$

$\Rightarrow \boxed{A=2} \Rightarrow \boxed{B=-1} \Rightarrow \boxed{C=3}$

$= \int\left(\frac{2}{x} + \frac{-x+3}{x^2+4x+5}\right)dx = 2\ln|x| + \int \frac{-x+3}{(x+2)^2+1}\,dx$

$u = x+2 \Rightarrow \begin{cases} x = u-2 \\ -x = 2-u \\ -x+3 = 5-u \end{cases}$

$= 2\ln|x| + \int \frac{5-u}{1+u^2}\,du$

$= 2\ln|x| + 5\int \frac{1}{1+u^2}\,du - \int \frac{u}{1+u^2}\,du$

$t = 1+u^2$

$dt = 2u\,du$

$= 2\ln|x| + 5\arctan(u) - \frac{1}{2}\int \frac{dt}{t}$

$= \ln x^2 + 5\arctan(x+2) - \frac{1}{2}\ln\left(1+(x+2)^2\right) + C$

Question 3 (10+5=15 points)

a) Determine the linearization of $f(x) = 2 + \int_1^{x^4} \sec^{37}(t-1)dt$ at $x = -1$.

$f(x) = 2 + g(x^4)$ where $g(x) = \int_1^x \sec^{37}(t-1)\,dt$

By chain rule, we have $f'(x) = g'(x^4) \cdot 4x^3$ where

$g'(x) = \frac{d}{dx} \int_1^x \sec^{37}(t-1)dt = \sec^{37}(x-1)$ [by Fundamental Thm of Calculus]

Thus $g'(x^4) = \sec^{37}(x^4-1)$

so $f'(x) = \sec^{37}(x^4-1) \cdot 4x^3$.

On the other hand $f'(-1) = -g'(1)\,4 = -4\sec^{37} 0 = \frac{-4}{\cos^4 0} = \frac{-4}{1} = -4$

$f(-1) = 2 + \int_1^{(-1)^4} \sec^4(t-1)\,dt = 2 + 0 = 2$

Thus

Linearization of f at $x=-1$ is

$L(x) = f(-1) + f'(-1)(x-(-1)) = 2 + (-4)(x+1)$

$= -4x - 2$

b) Using part (a) approximate $f(-0.99)$.

By linear approximation at $x=-1$, If x is close to -1,

then $f(x) \approx L(x)$

$x = -0.99$ is close to -1, thus

$f(-0.99) \approx L(-0.99) = +4 \cdot \frac{+99}{100} - 2 = \frac{396-200}{100} = \frac{196}{100}$

$= 1{,}96$

Question 4 (15 points) Find the absolute minimum and absolute maximum points and values of $f(x) = (x^3 - 9x^2 + 15x)^{2/3}$ on $[-1, 6]$.

Since $f(x)$ is continuous on the close interval $[-1, 6]$, by Extreme value theorem, absolute minimum and absolute maximum points of f on $[-1, 6]$ exist.

$$f'(x) = \frac{2}{3}(x^3 - 9x^2 + 15x)^{-1/3}(3x^2 - 18x + 15) = \frac{2}{3}\frac{3x^2 - 18x + 15}{\sqrt[3]{x^3 - 9x^2 + 15x}}$$

Note that when $x^3 - 9x^2 + 15x = 0$, that is

when $x(x^2 - 9x + 15) = 0$, that is

when $x = 0$ or $x = \frac{9 \mp \sqrt{21}}{2}$

$f'(x)$ is not defined

Note that $x = 0$ & $x = \frac{9 - \sqrt{21}}{2}$ are in $(-1, 6)$

but $x = \frac{9 + \sqrt{21}}{2} \notin (-1, 6)$. Thus

$x = 0$ & $x = \frac{9 - \sqrt{21}}{2}$ are singular pts in $(-1, 6)$ and

$\boxed{f(0) = 0}$, $\boxed{f\left(\frac{9 - \sqrt{21}}{2}\right) = 0}$

critical pts: $f'(x) = 0 \Rightarrow 3x^2 - 18x + 15 = 0$

$\Rightarrow (x-5)(x-1) = 0 \Rightarrow x = 1$ & $x = 5$ are critical pts inside $(-1, 6)$

$\boxed{f(1) = 7^{2/3}}$ $\boxed{f(5) = 25^{2/3}}$

End pts: $\boxed{f(-1) = 25^{2/3}}$, $\boxed{f(6) = 18^{2/3}}$

Therefore $f(-1) = f(5) = 25^{2/3}$ is absolute max. value, $x = -1$, $x = 5$ are abs max. points

and

$f(0) = f\left(\frac{9 - \sqrt{21}}{2}\right) = 0$ is absolute min. value, $x = 0$, $x = \frac{9 - \sqrt{21}}{2}$ are abs. min. points

Name,Surname: .... ***Student Id:*** .... ***Signature:*** ......

Question 5 (4+12=16 points) Let D be the region on the right half plane $(x \geq 0)$ which lies below the line $y = x + 1$ and inside the ellipse $x^2 + \dfrac{y^2}{4} = 1$.

a) Sketch the region D.

$y = x+1$ & $x^2 + \frac{y^2}{4} = 1$

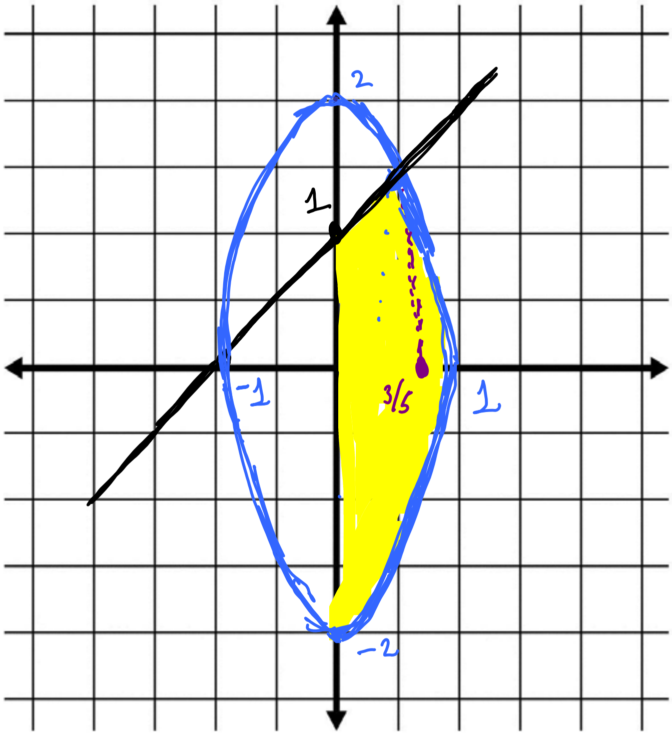
$\Rightarrow x^2 + \frac{(x+1)^2}{4} = 1$

$\Rightarrow 4x^2 + x^2 + 2x + 1 = 4$

$\Rightarrow 5x^2 + 2x - 3 = 0$

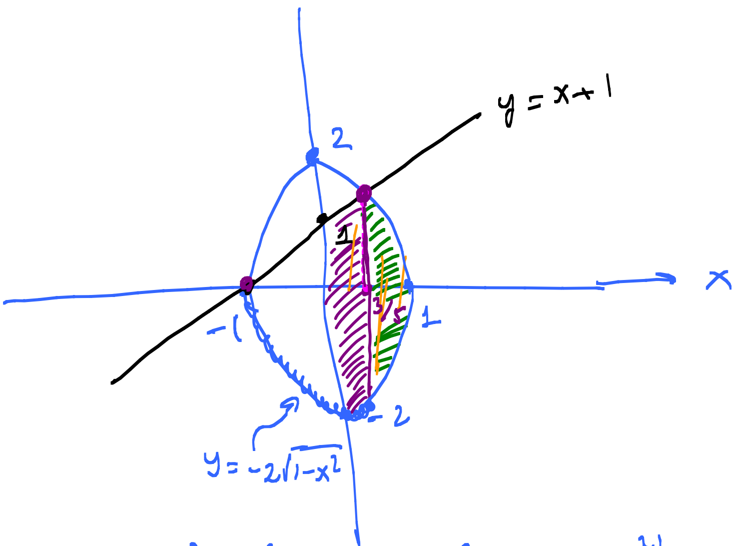
$\Rightarrow x = \frac{-2 \mp 8}{10}$

$\Rightarrow x_1 = -1, \quad x_2 = 3/5$



b) Express the area of D as a definite integral or sum of definite integrals. (Do **not** evaluate integral(s))

$$\text{Area} = \int_0^{3/5} \left[(x+1) - (-2\sqrt{1-x^2}) \right] dx + 2\int_{3/5}^{1} (2\sqrt{1-x^2})\, dx$$



$x^2 + \frac{y^2}{4} = 1 \Rightarrow y^2 = 4(1-x^2)$

$\Rightarrow y = \mp 2\sqrt{1-x^2}$

Thus

$$A = \int_0^{3/5} (x + 1 + 2\sqrt{1-x^2})\, dx + 4\int_{3/5}^{1} \sqrt{1-x^2}\, dx$$

Ok

$$\int_{-2}^{1} \sqrt{1-\frac{y^2}{4}}\, dy + \int_1^{8/5} \left[\sqrt{1-\frac{y^2}{4}} - (y-1) \right] dy$$

Question 6 (3+5+5+5=18 points) Let $f(x) = \dfrac{x^3 - 3x^2 + 1}{x^3}$

a) Find (if any) all horizontal and vertical asymptotes of f.

H.A.: $\lim_{x \to +\infty} f(x) = 1 < \infty \Rightarrow y = 1$ is a H.A.

$\lim_{x \to -\infty} f(x) = 1 < \infty \Rightarrow y = 1$ is a H.A.

$\Big\}$ $y = 1$ is the only H.A.

V.A.: $\lim_{x \to 0^+} f(x) = +\infty \Rightarrow x = 0$ is a V.A.

$\lim_{x \to 0^-} f(x) = -\infty \Rightarrow$ " " " " "

$\Big\}$ $x = 0$ is the only V.A.

b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Determine (if any) local minimum and local maximum of f.

$f'(x) = \left[(3x^2 - 6x)x^3 - 3x^2(x^3 - 3x^2 + 1) \right] / x^6$

$= \dfrac{3(x^2-1)}{x^4}$ defined $\forall x \in \text{Domain}(f) = \mathbb{R} \setminus \{0\}$

$\Rightarrow$ no singular pts.

$f'(x) = 0 \Rightarrow x = \mp 1$ are critical pts.

x		-1		0		$+1$	
$f'(x)$	$+$	0	$-$	$\Vert$	$-$	0	$+$
	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

- f is increasing on $(-\infty, -1]$ and on $[1, \infty)$
- " " decreasing " $[-1, 0)$ " " $(0, 1]$
- $x = -1$ is a local MAX point
- $x = +1$ is a local min point

c) Find the intervals on which f is concave up and the intervals on which f is concave down. Determine (if any) all inflection points of f.

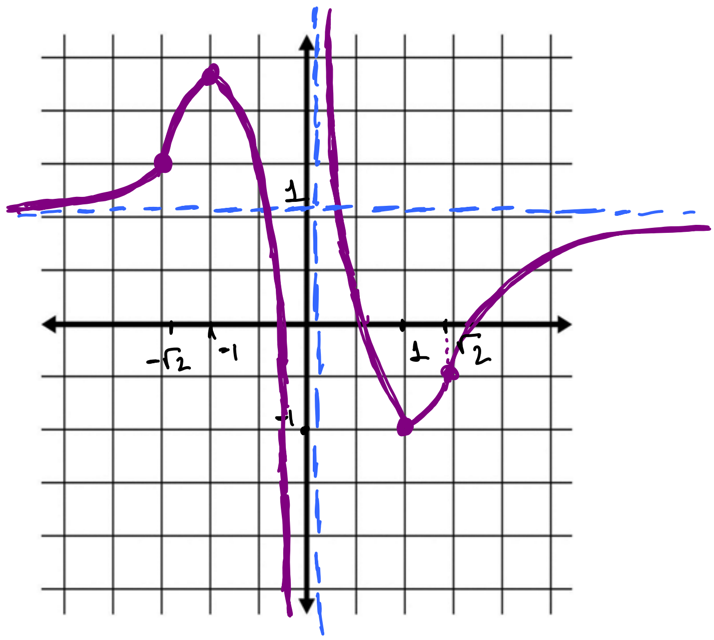
$f''(x) = 3\left[2x \cdot x^4 - 4x^3(x^2-1) \right] / x^8 = \dfrac{3}{x^5}(4 - 2x^2)$

x		$-\sqrt{2}$		0		$\sqrt{2}$	
$f''(x)$	$+$	0	$-$	$\Vert$	$+$	0	$-$
$f(x)$	$\cup$		$\cap$		$\cup$		$\cap$

- f is concave up on $(-\infty, -\sqrt{2})$ and on $(0, \sqrt{2})$
- " " " down " $(-\sqrt{2}, 0)$ and on $(\sqrt{2}, \infty)$
- Inflection points of f are $x = \mp\sqrt{2}$

d) Sketch the graph of f on the given coordinate plane.

x		$-\sqrt{2}$		-1		0		1		$\sqrt{2}$	
f'	$+$		$+$	0	$-$	$\Vert$	$-$	0	$+$		$+$
f''	$+$	0	$-$		$-$	$\Vert$	$+$		$+$	0	$-$
f	$\nearrow\cup$		$\nearrow\cap$		$\searrow\cap$		$\searrow\cup$		$\nearrow\cup$		$\nearrow\cap$



M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY Final Exam	

Code : *Math 119* Acad. Year : *2014-2015* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *January 08.2015* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (6+6=12 points) Determine whether the following integrals are convergent or divergent. Either compute the improper integrals directly or use comparison test, (limit comparison test is NOT allowed).

a) $\displaystyle\int_0^3 \frac{(1+e^{-x})(x^3+x+1)}{x^5}\,dx.$ $= \int_0^3 f(x)\,dx$ is improper at $x=0$

Note that $1+e^{-x} > 1$ $\forall x$ and on $(0,3]$ we have

$f(x) > \frac{1}{x^5}$.

Thus $\displaystyle\int_0^3 f(x)\,dx \geqslant \underbrace{\int_0^3 \frac{1}{x^5}\,dx}_{\text{divergent by p-test } p=5>1} \cancel{\geq} 0$

OR $\displaystyle = \lim_{c\to 0^+} \int_c^3 x^{-5}\,dx = \lim_{c\to 0^+} \frac{x^{-4}}{-4}\Big|_c^3$

$\displaystyle = -\frac{1}{4}\lim_{c\to 0^+}\left[\frac{1}{3^4} - \frac{1}{c^4}\right] = +\infty$ divergent

$\frac{1}{c^4} \to \infty$

So $\int_0^3 f(x)\,dx$ is divergent by comparison test (note since $f(x)>0$ $\forall x\in(0,3]$ we can use comparison test)

b) $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^2}\,dx.$ This integral is improper at $x=1$ and at ∞. First split the integral into two integrals, each one of which has only one improperness at one of its boundary: $\int_1^e f(x)\,dx$, $\int_e^\infty f(x)\,dx$

We know that given improper integral is convergent if and only if both of these improper integrals is convergent.

First consider $\displaystyle\int_1^e f(x)\,dx = \lim_{c\to 1^+}\int_c^e \frac{1}{x\ln^2 x}\,dx = \lim_{c\to 1^+}\int_{\ln c}^1 \frac{1}{u^2}\,du$ divergent by p-test

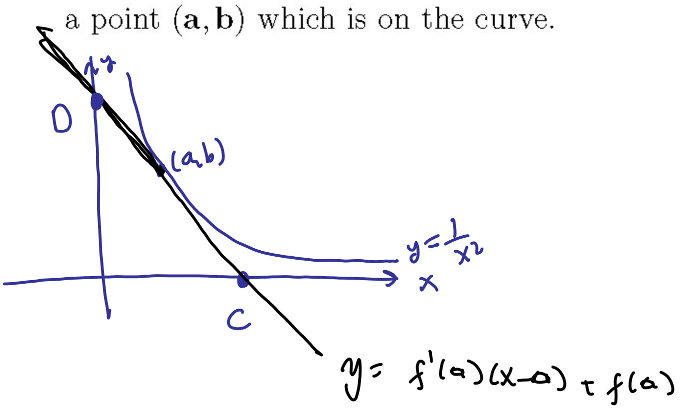
$u=\ln x$

$du = \frac{1}{x}\,dx$

Since this improper integral is divergent, we don't need to check the other one, given integral $\int_1^\infty f(x)\,dx$ is divergent

Question 2 (3+3+10=16 points)

a) Write down the equation of the tangent line to the curve $y = \dfrac{1}{x^2}$ for $x > 0$ through a point $(\mathbf{a}, \mathbf{b})$ which is on the curve.



$$y = f(x) = \frac{1}{x^2} \Rightarrow f'(x) = \frac{-2}{x^3} \Rightarrow f'(a) = \frac{-2}{a^3}$$

$$y = \frac{-2}{a^3}(x-a) + \frac{1}{a^2}$$

b) Let C and D be the points where the tangent line in part (a) intersects the x-axis and the y-axis. Write the coordinates of C and D in terms of the variable $\mathbf{a}$.

$$y = \frac{-2}{a^3}(x-a) + \frac{1}{a^2}$$

$$x = 0 \Rightarrow y = \frac{2a}{a^3} + \frac{1}{a^2} = \frac{3}{a^2} \Rightarrow D = (0, 3/a^2)$$

$$y = 0 \Rightarrow \frac{2}{a^{\cancel{3}}}(x-a) = \frac{1}{\cancel{a^2}} \Rightarrow 2x - 2a = a \Rightarrow x = \frac{3}{2}a$$

$$\Rightarrow C = \left(\tfrac{3}{2}a, 0\right)$$

c) For which point $(\mathbf{a}, \mathbf{b})$ the distance between C and D is minimum?

$$d(a) = \sqrt{\left(0 - \tfrac{3}{2}a\right)^2 + \left(\tfrac{3}{a^2} - 0\right)^2}, \quad a \in (0, \infty)$$

to find minimum of $d(a)$, first find min. of $g(a) = d^2(a)$ and then take square-root.

$$g(a) = \frac{9}{4}a^2 + \frac{9}{a^4} \; ; \quad a \in (0, \infty).$$

Note that $\lim_{a \to 0^+} g(a) = +\infty$ and $\lim_{a \to \infty} g(a) = +\infty$ and $g(a)$ is diff'ble on $(0,\infty)$

Hence $g(a)$ has absolute min on $(0,\infty)$ and it occurs at critical pt.

$$g'(a) = \frac{9}{2}a - \frac{36}{a^5} = 0 \Rightarrow \frac{9}{2}a = \frac{36}{a^5} \Rightarrow a^6 = \frac{\cancel{36}^{\,4} \cdot 2}{\cancel{9}} = 2^3 \Rightarrow a = \sqrt{2}$$

is the only critical point

Thus absolute min of $g(a)$, and hence $d(a)$ occurs when $a = \sqrt{2}$

$$\therefore (a, b) = \left(\sqrt{2}, \tfrac{1}{2}\right)$$

Question 3 (5+5+5=15 points)

Let f be the function defined by $f(x) = \begin{cases} (e-2)x^2 & \text{if } x \leq 0 \\ \dfrac{e^x - x - 1}{x} & \text{if } x > 0 \end{cases}$

a) Is f a continuous function?

Except $x=0$, f is obviously continuous

at $x=0$: • $f(0) = (e-2).0^2 = 0$

• $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (e-2)x^2 = 0$

• $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x - x - 1}{x}$ $(\frac{0}{0}$ type$)$ $\overset{\text{L'Hop's R.}}{=} \lim_{x\to 0^+} e^x - 1 = 0$

$\lim_{x\to 0} f(x) = f(0) = 0$ $\therefore$ f is cont. at $x=0$

$\therefore$ f is cont. $\forall x$

b) Is f a differentiable function?

Except $x=0$, f is obviously differentiable

at $x=0$

$\lim_{h\to 0^-} \frac{f(h) - \overbrace{f(0)}^{=0}}{h} = \lim_{h\to 0^-} \frac{(e-2)h^{\cancel{2}}}{\cancel{h}} = 0$

$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{e^h - h - 1}{h^2}$ $(\frac{0}{0}$ type$)$ $= \frac{1}{2}$ by L'Hop's Rule

$0 \neq \frac{1}{2}$ $\therefore$ f is not diff'le at $x=0$

consider $\lim_{h\to 0^+} \frac{e^h - 1}{2h}$ $(\frac{0}{0}$ type$)$ $= \frac{1}{2}$ by L'Hop's Rule

consider $\lim_{h\to 0} \frac{e^h}{2} = \frac{1}{2}$ exists

c) Is the following statement True or False? Explain your answer.

There exists a point $c \in (-1, 1)$ *such that* $f'(c) = \dfrac{f(1) - f(-1)}{2}$.

$f(1) = e-2$, $f(-1) = e-2$ $\Rightarrow$ $\frac{f(1) - f(-1)}{2} = 0$

on $(-1,0)$ we have $f'(c) = 2c(e-2) = 0 \Rightarrow c = 0 \notin (-1,0)$

on $(0,1)$ we have $f'(c) = \frac{(e^c - \cancel{1})c - (e^c - \cancel{c} - 1)}{c^2} = \frac{e^c(c-1)+1}{c^2} = 0$

$\Rightarrow \underbrace{e^c(c-1)}_{g(c)} = -1$

Note that $g(0) = -1$ & $g'(c) = e^c(c-1) + e^c = ce^c > 0$ on $c \in (0,1)$

so $g(x)$ is increasing on $(0,1)$. Thus $g(c) \neq -1$ $\forall c \in (0,1)$

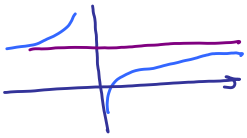
$\therefore \nexists c \in (0,1)$ s.t. $g(c) = -1$.

$\therefore$ There is no such $c \in (-1,1)$. This is not suprising since f is not diff'le at $x = 0 \in (-1,+1)$.

(if f were diff'le on $(-1,+1)$, then since f is also continuous on $[-1,+1]$, given statement would be true by Mean Value Theorem)

Question 4 (14 points)

Find all (if any) horizontal and vertical asymptotes of $f(x) = (1+\sqrt{x})^{1/x}$.



$y = c$ is H.A. if $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$

$$\lim_{x\to+\infty} f(x) = e^{\lim_{x\to+\infty} \frac{\ln(1+\sqrt{x})}{x}} \overset{\text{L'Hôpital}}{=} e^{\lim_{x\to+\infty} \frac{1}{2\sqrt{x}(1+\sqrt{x})}} = e^0 = 1 \Rightarrow y = 1 \text{ Horizontal Asymp.}$$

since Domain of f is $(0,\infty)$ we don't need to check $\lim_{x\to-\infty} f(x)$.

$x = c$ is V.A. if $\lim_{x\to c} f(x) = \mp\infty$

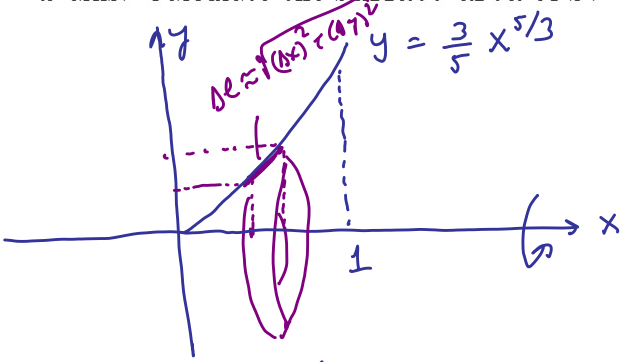
candidate: $x = 0$: check: $\lim_{x\to 0^+} f(x) = e^{\lim_{x\to 0^+} \frac{\ln(1+\sqrt{x})}{x}}$ ($\frac{0}{0}$ type)

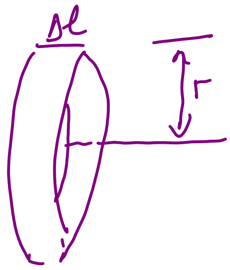
$$= e^{\lim_{x\to 0^+} \boxed{\frac{1}{2\sqrt{x}(1+\sqrt{x})}} \to \infty} = \infty$$

Thus $x = 0$ is Vertical Asymptote

Question 5 (15 points)

Let S be the surface obtained by rotating the curve $y = \dfrac{3}{5}x^{5/3}$, $0 \le x \le 1$ about the x-axis. Calculate the **surface area** of S.





$\Delta SA \approx 2\pi r \,\Delta\ell$

$= 2\pi \cdot y(x) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$

$y'(x) = \frac{3}{5} \cdot \frac{5}{3} x^{2/3} = x^{2/3}$

$$SA = 2\pi \int_0^1 \frac{3}{5} x^{\frac{5}{3}} \sqrt{1 + x^{4/3}}\, dx = 2\pi \, \frac{3}{5} \, \frac{3}{4} \int_1^2 u^{1/2}(u-1)\, du$$

$u = 1 + x^{4/3}$

$du = \frac{4}{3} x^{\frac{1}{3}} dx$

$$= \frac{9\pi}{10} \int_1^2 u^{3/2} - u^{1/2}\, du = \frac{9\pi}{10}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^2$$

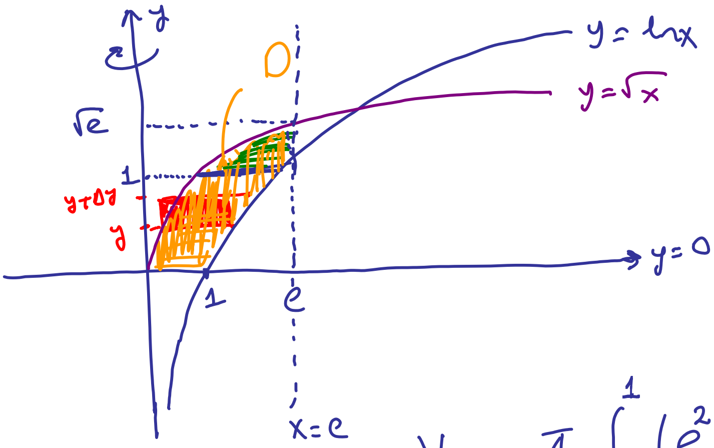
$$= \frac{9\pi}{5}\left[\left(\frac{2^{5/2}}{5} - \frac{2^{3/2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right] = \frac{9\pi}{5}\left[\frac{2\sqrt{2}}{15} + \frac{2}{15}\right] = \frac{6\pi(1+\sqrt{2})}{25}$$

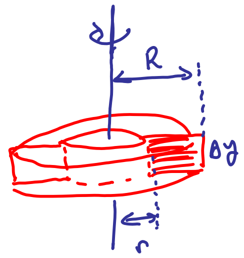
Name,Surname: .... **Student Id:** .... **Signature:** ......

Question 6 (6+6+6=18 points)

Let D be the region bounded by the curves $y=\sqrt{x}$, $y=\ln x$, $y=0$ and $x=e$.

a) Express (DO NOT EVALUATE) the volume of the solid S obtained by rotating the region D about y-axis as an integral(s) by using slicing (disk/washer) method.

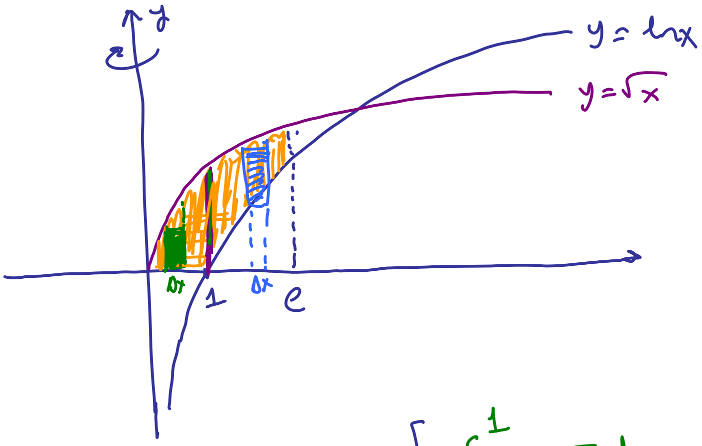




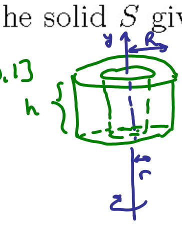
$$\Delta V = \pi(R^2-r^2)\,\Delta y$$
$$= \pi\left[(e^y)^2-(y^2)^2\right]\Delta y$$

$$V = \pi\int_0^1 (e^{2y}-y^4)\,dy + \pi\int_1^{\sqrt{e}} (e^2-y^4)\,dy$$

b) Express (DO NOT EVALUATE) the volume of the solid S given in part (a) as an integral(s) by using cylindrical shells method.



on $[0,1]$



$\Delta V = \pi(R^2-r^2)h$

$R = x+\Delta x$

$r = x$

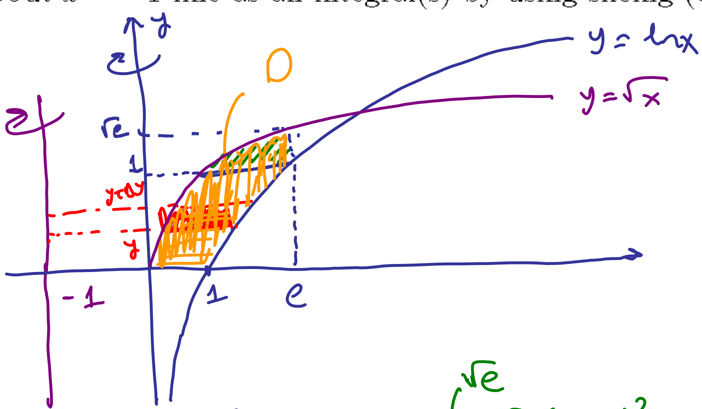
$R^2-r^2 = \cancel{x^2}+2x\Delta x+\underbrace{(\Delta x)^2}_{\approx 0}-\cancel{x^2}$

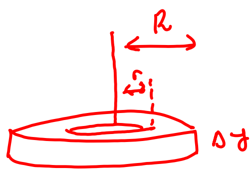
$\Delta V = 2\pi x\,\Delta x\,\sqrt{x}$

on $[1,e]$ $\quad \Delta V = 2\pi x(\sqrt{x}-\ln x)\,\Delta x$

$$V = 2\pi\left[\int_0^1 x\sqrt{x}\,dx + \int_1^e x(\sqrt{x}-\ln x)\,dx\right]$$

c) Express (DO NOT EVALUATE) the volume of the solid S obtained by rotating the region D about $x=-1$ line as an integral(s) by using slicing (disk/washer) method.





$R = 1+e^y$

$r = 1+y^2$

$$\Delta V = \pi\left[(1+e^y)^2-(1+y^2)^2\right]\Delta y$$

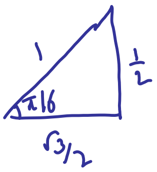
$$V = \pi\int_0^1\left[(1+e^y)^2-(1+y^2)^2\right]dy + \pi\int_1^{\sqrt{e}}\left[(1+e)^2-(1+y^2)^2\right]dy$$

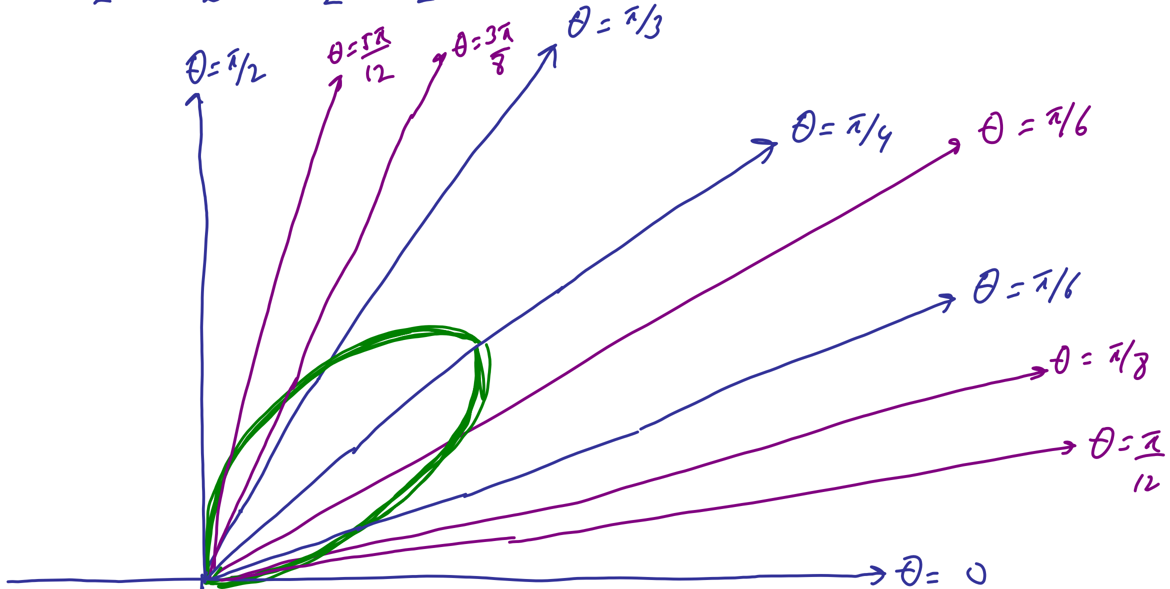
Question 7 (5+5=10 points)

Let C be the curve given by the polar equation $r = \sin 2\theta$.

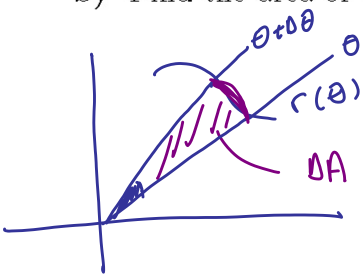
a) Sketch the graph of the part of the curve C **in the first quadrant** (that is, where $x \geq 0, y \geq 0$).

θ	0	$\pi/12$	$\pi/8$	$\pi/6$	$\pi/4$	$\pi/3$	$3\pi/8$	$\frac{5\pi}{12}$	$\pi/2$
2θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$





b) Find the area of the region bounded by the curve sketched in part (a).



$$\Delta A \approx \frac{\cancel{\pi}\, r^2(\theta)}{2\cancel{\pi}} \Delta\theta \Rightarrow A = \frac{1}{2}\int_0^{\pi/2} r^2(\theta)\, d\theta$$

$$A = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta = \frac{1}{2\cdot 2}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{4}\frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\, dx$$

$x = 2\theta$

$dx = 2d\theta$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \pi/8$$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	

Code : *Math 119* Acad. Year : *2013-2014* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *November.23.2013* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (12 pts) Find the derivative $\frac{dy}{dx}$ for the following functions.

a) $y = \sqrt{\frac{\sin(x^2)}{\tan x}} = \left(\frac{\sin(x^2)}{\tan x}\right)^{1/2}$

$$y' = \frac{1}{2}\left(\frac{\sin(x^2)}{\tan x}\right)^{-1/2}\left[\frac{2x\cos(x^2)\tan x - \sin(x^2)\sec^2 x}{\tan^2 x}\right]$$

or/ $\ln y = \frac{1}{2}\left[\ln(\sin(x^2)) - \ln(\tan x)\right]$ take derivative of both sides

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{2x\cos(x^2)}{\sin(x^2)} - \frac{\sec^2 x}{\tan x}\right] \Rightarrow y' = \frac{1}{2}\left(\frac{\sin(x^2)}{\tan x}\right)\left[2x\cot(x^2) - \frac{1}{\cos x \cdot \sin x}\right]$$

b) $y = (x + x^2)^{\ln x}$

$\Rightarrow \ln y = \ln x \cdot \ln(x+x^2)$ take derivatives of both sides:

$$\Rightarrow \frac{y'}{y} = \frac{1}{x}\ln(x+x^2) + \ln x \cdot \frac{(1+2x)}{x+x^2}$$

$$y' = (x+x^2)^{\ln x} \cdot \left[\frac{\ln(x+x^2)}{x} + \ln x \cdot \frac{(1+2x)}{1+x}\right]$$

or/ $\ln y = \ln x \cdot \ln(x+x^2) \Rightarrow y = e^{\ln x \cdot \ln(x+x^2)} \Rightarrow y' = ---$

Question 2 (6 pts) Find the derivative $f'(a)$ in terms of g if $f(x) = (x-a)g(x)$ where $g(x)$ is continuous at $x = a$. (*Caution: g may not be differentiable at $x = a$*).

First note that $f(a) = (a-a)g(a) = 0$.

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{(x-a)g(x)}{(x-a)} = \lim_{x\to a} g(x) = g(a)$$ since g is continuous at $x = a$.

since above limit exists, $f'(a)$ exists and $f'(a) = g(a)$

Question 3 (20 pts) Evaluate the following limits. *(Do **NOT** use L'Hospitals Rule).*

a) $\lim_{x\to\infty} \frac{\cos(x\sin x)}{x^2}$

$-1 \leq \cos(x\sin x) \leq +1 \quad \forall x \in \mathbb{R}$, and hence

$\frac{-1}{x^2} \leq \frac{\cos(x\sin x)}{x^2} \leq \frac{1}{x^2} \quad \forall x \neq 0.$

since $\lim_{x\to\infty} \frac{-1}{x^2} = 0 = \lim_{x\to\infty} \frac{1}{x^2}$, by squeezing thm, we have

$\lim_{x\to\infty} \frac{\cos(x\sin x)}{x^2} = 0.$

b) $\lim_{x\to 0} \frac{|\sin x|}{x}$

$$\left.\begin{array}{l} \lim_{x\to 0^+} \frac{|\sin x|}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \\ \lim_{x\to 0^-} \frac{|\sin x|}{x} = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1 \end{array}\right\} \Rightarrow \lim_{x\to 0} \frac{|\sin x|}{x} \text{ does not exist}$$

Muhiddin UGUZ

c) $\lim_{x\to\infty} \frac{x^3 + \sqrt{x^6 + 2x^5 + 1}}{x^3 + 2x^2 + x + 1}$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(1 + \sqrt{1 + \frac{2}{x} + \frac{1}{x^6}}\right)}{\cancel{x^3}\left(1 + \frac{2}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)} = 2$$

with $\frac{2}{x}\to 0$, $\frac{1}{x^6}\to 0$, $\frac{2}{x}\to 0$, $\frac{1}{x^2}\to 0$, $\frac{1}{x^3}\to 0$.

d) $\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3}$

multiply and divide by $\sqrt{1+\tan x} + \sqrt{1+\sin x}$ and use the identity $(a-b)(a+b) = a^2 - b^2$ to get;

$$\lim_{x\to 0} \frac{1 + \tan x - 1 - \sin x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} = \lim_{x\to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x^3} \frac{(1-\cos x)}{\cos x} \frac{1}{\sqrt{1+\tan x} + \sqrt{1+\sin x}} \cdot \frac{(1+\cos x)}{(1+\cos x)} =$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin x}{x}\right)^3}_{\to 1} \frac{1}{\underbrace{\cos x}_{\to 1} \cdot \underbrace{\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}_{\to 2} \underbrace{(1+\cos x)}_{\to 2}} = \frac{1}{4}$$

Question 4 (10 pts) A particle moves along the curve $y = 4x - x^3$ and it's x-coordinate is decreasing at a constant rate of $\frac{1}{3}$ units per second.

a) At what rate does the y-coordinate of the particle change when $x = 2$?

When time is equal to t, particle is at the point $(x(t), y(t))$ where $y(t) = 4x(t) - x^3(t)$. We are given that $\frac{dx}{dt}(t) = -\frac{1}{3}$ units/second.

and when $x(t) = 2$ we have $y(t) = 4 \cdot 2 - 2^3 = 0$.

Taking derivative of both sides of $y(t) = 4x(t) - x^3(t)$ with respect to t, we get $y'(t) = 4x'(t) - 3x^2(t)\,x'(t)$. Hence if $x(t) = 2$ we have

$$y'(t) = 4 \cdot \frac{-1}{3} - \cancel{3} \cdot 2^2 \cdot \frac{-1}{\cancel{3}} = \frac{-4}{3} + 4 = +\frac{8}{3} \text{ units/sec.}$$

b) At what rate does the slope of the tangent line change when $x = 2$?

$$\text{slope} = y' = \frac{dy}{dx} = 4 - 3x^2.$$

$$\frac{dy'}{dt} = -6\,x(t)\,x'(t) \Rightarrow \left.\frac{dy'}{dt}\right|_{x(t)=2} = -6 \cdot 2 \cdot \frac{-1}{3} = +4 \text{ units/sec}$$

Question 5 (10 pts) Consider $f(x) = \begin{cases} \dfrac{1-\cos x}{x} & \text{if } -\dfrac{\pi}{2} \le x < 0 \\ ax + b & \text{if } 0 \le x \le 1 \end{cases}$

Determine the values of a, b such that $f'(x)$ exists for all x with $-\frac{\pi}{2} < x < 1$.

Since $f'(x) = (x\sin x + \cos x - 1)/x^2$ if $-\pi/2 < x < 0$

$f'(x) = a$ if $0 < x < 1$, values of a & b are important only for differentiability of f at $x = 0$.

To be differentiable at $x = 0$, first of all f must be continuous at $x = 0$; that is

$$f(0) = b = \lim_{x \to 0} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1-\cos x}{x} = 0. \text{ Hence } \boxed{b = 0}$$

Moreover, to be diff'le, (Note $f(0) = a \cdot 0 = 0$).

We are given that $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$ exists.

Hence $\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{f(x)}{x} \Rightarrow \lim_{x \to 0^-} \frac{1-\cos x}{x^2} = \lim_{x \to 0^+} \frac{ax}{x} = a$

$$\Rightarrow a = \lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x^2\,(1+\cos x)} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} = \frac{1}{2}$$

$$\therefore (a, b) = \left(\frac{1}{2}, 0\right)$$

Question 6 (10 pts) Prove that the equation $\ln(x) = \frac{1}{x}$ for $x > 0$ has a unique solution. Explain.

Let $f(x) = \ln x - \frac{1}{x}$. Obviously $f(x)$ is continuous $\forall x > 0$.

Recall that $e \approx 2.71.. \Rightarrow 2 < e < 3 \Rightarrow \frac{1}{2} > \frac{1}{e} > \frac{1}{3} \Rightarrow -\frac{1}{2} < -\frac{1}{e} < -\frac{1}{3}$

$f(e) = \ln e - \frac{1}{e} = 1 - \frac{1}{e} > 1 - \frac{1}{2} > 0$

$f(\frac{1}{e}) = \ln e^{-1} - e = -1 - e < 0$

since f is continuous by Intermediate Value There is $r \in (\frac{1}{e}, e)$ such that $f(r) = 0$

$f'(x) = \frac{1}{x} + \frac{1}{x^2} \gneq 0 \ \forall x > 0$, hence $f(x)$ is strictly increasing on $(0, \infty)$ and hence r is unique.

(because if $\exists\, r_1 < r_2$ with $f(r_1) = 0 = f(r_2)$ then as f is cont. on $[r_1, r_2]$ and diff'ble on (r_1, r_2), by Rolle's Thm (or more generally by Mean Value Thm), $\exists\, x_0 \in (r_1, r_2)$ with $f'(x_0) = 0$ contradicting $f'(x) \gneq 0 \ \forall x > 0$)

Question 7 (12 pts) The function $y = f(x)$ is given implicitly by $\cos\left(\frac{\pi y}{x}\right) = \frac{x^2}{y} - 4$.

a)Find $\frac{dy}{dx}$ at $(3, 3)$.

Take derivative of both sides of the given equation with respect to x; substitute $x = y = 3$;

$$-\sin\left(\frac{\pi y}{x}\right)\left(\frac{\pi y'(x) \cdot x - \pi y}{x^2}\right) = \frac{2xy - x^2 y'}{y^2}$$

$$\underbrace{-\sin(\pi)}_{=0}(\cdots) = \frac{2 \cdot \cancel{3} \cdot \cancel{3} - \cancel{3^2} y'}{\cancel{3^2}} \Rightarrow 0 = 2 - y' \Rightarrow y' = 2$$

b)Find the linearization of $y = f(x)$ at $(3, 3)$ and use it to approximate $f(3.1)$.

$$y(3.1) = f(3.1) \approx f(3) + f'(3)(0.1) = y(3) + y'(3)\frac{1}{10}$$

$$= 3 + 2\frac{1}{10} = 3 + \frac{1}{5}$$

$$= \frac{16}{5} = 3.2$$

Last Name: . . . . . First Name: . . . . . Student Id: . . . . . Signature: . . .

Question 8 (20 points) Let $f(x) = \dfrac{e^x}{x-1}$

a) Find the domain, intercepts and all vertical and horizontal asymptotes of the graph of $f(x)$.

- Domain $= \mathbb{R} \setminus \{1\}$
- $x=0 \Rightarrow y = \dfrac{e^0}{0-1} = \dfrac{1}{-1} = -1 \Rightarrow (0,-1)$ - y intercept

  $y=0 \Rightarrow \dfrac{e^x}{x-1} = 0 \Rightarrow e^x = 0$ no solution $\Rightarrow$ no- x-intercept
- Horizontal Asym: $\lim\limits_{x\to+\infty} \dfrac{e^x}{x-1}$ $\left(\frac{\infty}{\infty} \text{ type}\right) \overset{\text{L'Hopital Rule}}{=} \lim\limits_{x\to\infty} e^x = \infty$

  $\lim\limits_{x\to-\infty} \dfrac{e^x}{x-1} = \lim\limits_{x\to-\infty} \underbrace{e^x}_{\to 0} \underbrace{\dfrac{1}{x-1}}_{\to 0} = 0 \Rightarrow y=0$ is HA
- Vertical Asymp: candidate is $x=1$: $\lim\limits_{x\to 1} \dfrac{e^x}{x-1} = \mp\infty \Rightarrow x=1$ is V.A.

b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Determine the local extreme values.

$f'(x) = \dfrac{e^x(x-1) - e^x}{(x-1)^2} = \dfrac{e^x(x-2)}{(x-1)^2}$. $\quad f'(x) = 0 \Rightarrow x=2$ is the only critical pt.

x		1		2	
f'	$-$	‖	$-$	0	$+$
f	↘	‖	↘		↗

$x=2$ is the only local minimum and hence absolute minimum. $f(2) = e^2$

f has no local (and hence absolute) maximum.

f is decreasing on $(-\infty, 1)$ and on $(1, 2)$

f is increasing on $(2, +\infty)$

c) Find the intervals on which f is concave up and the intervals on which f is down and the point(s) of inflection.

$f''(x) = \left[(e^x(x-2) + e^x)(x-1)^2 - 2(x-1)e^x(x-2)\right] / (x-1)^4$

$= \dfrac{e^x}{(x-1)^4}\left[(x-2+1)(x-1)^2 - 2(x-1)(x-2)\right] = \dfrac{e^x}{(x-1)^4}(x-1)\left[(x-1)^2 - 2(x-2)\right]$

$= \dfrac{e^x}{(x-1)^3}(x^2 - 2x + 1 - 2x + 4) = \dfrac{e^x}{(x-1)^3}\underbrace{(x^2 - 4x + 5)}_{\substack{b^2-4ac = 16-4\cdot 5 \\ = -4 < 0 \\ \text{has no root}}} \Rightarrow \operatorname{sign}(f'') = \operatorname{sign}(x-1)$

x		1	
f''	$-$	‖	$+$
f	∩		∪

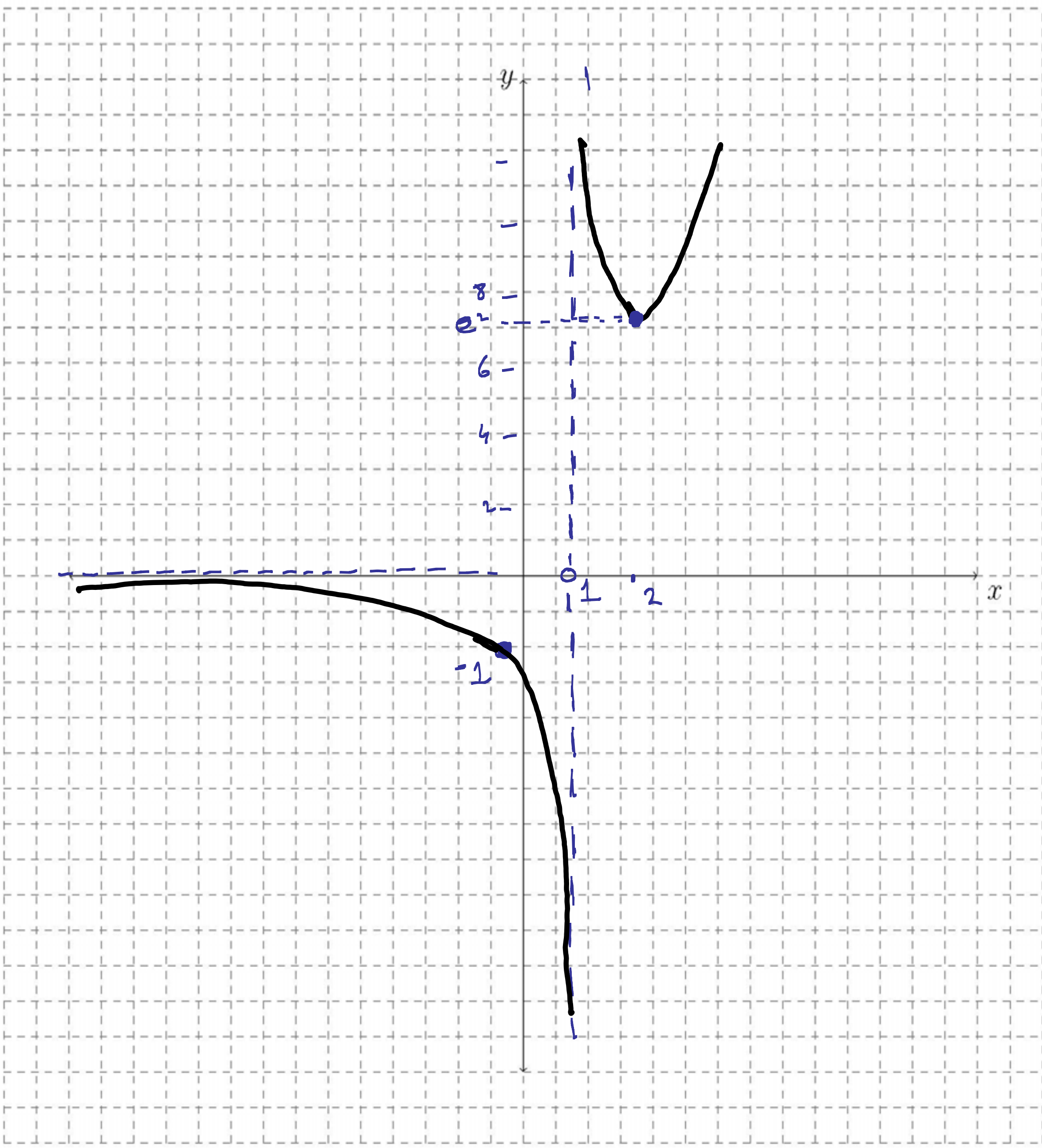
f is concave down on $(-\infty, 1)$

" " " up on $(1, \infty)$

since $1 \notin \text{Domain}(f)$, f has NO inflection point.

$f(2) = e^2 \approx 7.33$

d) Sketch the graph of $y = f(x)$.



M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 2	
Code : *Math 119* Acad. Year : *2013-2014* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *December 21,2013* Time : *9:30* Duration : *119 minutes*	Last, First Name Department : Signature : Student No. :	
	6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3-a-b	3-c-d	4	5	6	SHOW YOUR WORK	

Question 1 (8 pts) For the function $f(x) = xe^{(x^2)}$, write a Riemann Sum obtained by dividing the interval $[1, 2]$ into n equal length subintervals .

$\Delta x = \frac{2-1}{n} = \frac{1}{n}$

$P = \{ \underset{1}{\overset{\shortparallel}{x_0}}, \underset{1+1\frac{1}{n}}{\overset{\shortparallel}{x_1}}, x_2, \dots, \underset{2}{\overset{\shortparallel}{x_n}} \}$; $x_k = 1 + k\frac{1}{n}$

$\Delta x_i = x_{i+1} - x_i = \frac{1}{n}$

Let x_k^* be any point from the k^{th} subinterval $[x_{k-1}, x_k]$. Then

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} x_k^* e^{(x_k^*)^2} \cdot \frac{1}{n}$$

As an example let $x_k^* = x_k$ = right end point of the subinterval $[x_{k-1}, x_k]$. Then Right sum is

$$\sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n} \left(1+\frac{k}{n}\right) e^{\left(1+\frac{k}{n}\right)^2} \frac{1}{n}$$

Question 2 (8 pts) Evaluate $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\left(1 + \frac{2k}{n}\right)\cos\left((1 + \frac{2k}{n})^2\right)$.

Note that for $f(x) = x\cos(x^2)$, Riemann sum over the interval $[a,b]$ is given by $\sum_{k=1}^{n} f(a + k\Delta x).\Delta x$, where $\Delta x = \frac{b-a}{n}$

$x_0 = a = 1$, $\Delta x = \frac{2}{n}$ $\Rightarrow x_1 = 1 + 1\frac{2}{n}$, $x_2 = 1 + 2 \cdot \frac{2}{n}$

$\Rightarrow b = x_n = 1 + n\frac{2}{n} = 3$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\left(1+\frac{2k}{n}\right)\cos\left(\left(1+k\frac{2}{n}\right)^2\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\frac{b-a}{n}}_{\Delta x} \underbrace{(1+k\Delta x)\cos\left((1+k\Delta x)^2\right)}_{f(1+k\Delta x)}$$

where $f(x) = x\cos x^2$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(1+k\Delta x)\Delta x$$

$$= \int_1^3 f(x)\,dx = \int_1^3 x\cos(x^2)\,dx = \frac{1}{2}\int_1^9 \cos(u)\,du = \frac{1}{2}\sin(u)\Big|_1^9$$

$u = x^2$
$du = 2x\,dx$

$$= \frac{1}{2}(\sin 9 - \sin 1)$$

Question 3 **(8+8+8+8 pts)** Evaluate the following integrals.

a) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int \frac{\sin^2 x}{\sqrt{\cos x}} \sin x\,dx = \int \frac{(1-\cos^2 x)}{\sqrt{\cos x}} \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\int u^{-1/2}(1-u^2)\,du = \int u^{3/2} - u^{-1/2}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{1}{2}} + C = \frac{2}{5}\cos^{5/2}x - 2\cos^{1/2}x + C$$

b) $\displaystyle\int_1^2 \sqrt{\frac{1+\sqrt{x}}{x}}\,dx = \int_1^2 \frac{1}{\sqrt{x}}\sqrt{1+\sqrt{x}}\,dx = 2\int_2^{1+\sqrt{2}} \sqrt{u}\,du = 2\,\frac{2}{3}u^{3/2}\Big|_2^{1+\sqrt{2}}$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= \frac{4}{3}\left[(1+\sqrt{2})^{3/2} - 2^{3/2}\right]$$

c) $\displaystyle\int_{-1}^{1} \frac{\sin x}{1+x^2}\,dx$

Let $f(x) = \frac{\sin x}{1+x^2}$. Then $f(-x) = \frac{\sin(-x)}{1+(-x)^2} = -\frac{\sin x}{1+x^2} = -f(x)$

That is $f(x)$ is an odd function which is continuous on $[-1,1]$.

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx = -\int_1^0 f(-u)\,du + \int_0^1 f(x)\,dx$$

$u = -x$

$du = -dx$

$$= \int_0^1 f(-u)\,du + \int_0^1 f(x)\,dx$$

$$= -\int_0^1 f(u)\,du + \int_0^1 f(x)\,dx = 0$$

d) $\displaystyle\int \frac{\ln(\ln x)}{x\ln x}\,dx = \int \frac{\ln u}{u}\,du = \int w\,dw = \frac{w^2}{2} + C$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$w = \ln u$

$dw = \frac{1}{u}\,du$

$$= \frac{\ln^2 u}{2} + C$$

$$= \frac{\ln^2(\ln x)}{2} + C$$

Question 4 (8+8 pts) Let f be the function given by $f(x) = \int_0^{x^2} \cos(t^2)dt$.

a) Find $f'(x)$

$$\frac{d}{dx}\int_{x_0}^{b(x)} f(t)\,dt \underset{\left(\substack{\text{Let } u=b(x)\\ \frac{du}{dx}=b'(x)}\right)}{=} \frac{d}{dx}\left(\int_{b(x_0)}^{u} f(t)\,dt\right) \underset{\substack{\text{Chain}\\ \text{Rule}}}{=} \frac{d}{du}\left(\int_{b(x_0)}^{u} f(t)\,dt\right)\cdot\frac{du}{dx}$$

$$\underset{\uparrow}{=} f(u)\cdot u'(x)$$

Fundamental Thm of Calculus.

Hence $f'(x) = \cos(x^4)\cdot 2x$

b) Compute $\lim_{x\to 0} \frac{xf(x)}{\sin^2 x}$

$$(*) = \lim_{x\to 0} \frac{x\int_0^{x^2}\cos(t^2)dt}{\sin^2 x} = \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{\int_0^{x^2}\cos(t^2)dt}{\sin x}$$

- $\lim_{x\to 0}\frac{x}{\sin x} = \boxed{1}$
- $\lim_{x\to 0}\frac{\int_0^{x^2}\cos(t^2)dt}{\sin x}$ $\left(\frac{0}{0}\text{ type}\right)$ $\underset{\substack{\uparrow\\ \text{L'Hop's Rule}}}{=} \lim_{x\to 0}\frac{\cos(x^4)\cdot 2x}{\cos x} = \frac{1\cdot 0}{1} = \boxed{0}$

since each limit exists

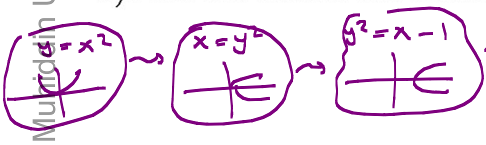
$$= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\cdot\left(\lim_{x\to 0}\frac{\int_0^{x^2}\cos(t^2)dt}{\sin x}\right) = 1\cdot 0 = 0$$

$(*)$ OR $= \lim_{x\to 0}\frac{f(x)/x}{\sin^2 x/x^2} = f'(0) = 0$ (since $f(0)=0$) by part (a)

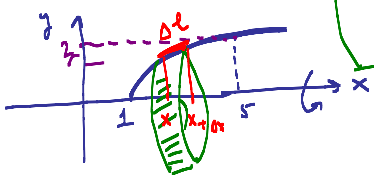
$$= \lim_{x\to 0}\frac{\frac{f(x)-f(0)}{x-0}}{\frac{\sin^2 x}{x^2}} = \frac{0}{1} = 0 \qquad \left(\frac{\sin^2 x}{x^2}\to 1\right)$$

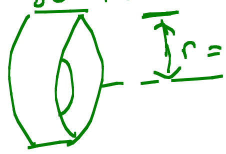
Question 5 (8+8 pts) Consider the curve C given by $y = \sqrt{x-1}$ where $1 \le x \le 5$.

a) Find the area of the surface obtained by rotating the curve C about x-axis.



$y=\sqrt{x-1} \Rightarrow y^2 = x-1$ & $y \ge 0$.





$\Delta\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$r = f(x) = \sqrt{x-1}$

OR
$$SA = 2\pi\int_0^2 y\sqrt{1+4y^2}\,dy = \frac{\pi}{6}\left[17^{3/2}-1\right]$$

$$\Delta SA = 2\pi r\,\Delta\ell = 2\pi f(x)\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

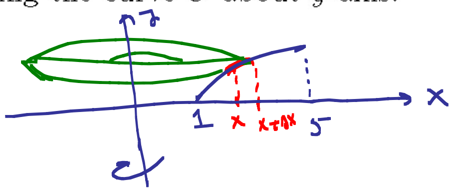
$f(x) = \sqrt{x-1} \Rightarrow f'(x) = \frac{1}{2\sqrt{x-1}}$

$$SA = 2\pi\int_1^5 \sqrt{x-1}\cdot\sqrt{1+\frac{1}{4(x-1)}}\,dx = 2\pi\int_1^5\sqrt{(x-1)+\frac{1}{4}}\,dx =$$

$$= 2\pi\int_1^5\sqrt{x-\frac{3}{4}}\,dx \underset{\substack{u=x-3/4\\ du=dx}}{=} 2\pi\int_{1/4}^{17/4} u^{1/2}\,du = 2\pi\frac{2}{3}u^{3/2}\Big|_{1/4}^{17/4} = \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]$$

$$= \frac{\pi}{6}\left[17^{3/2}-1\right]$$

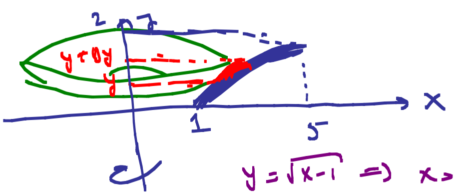
b) Write (do **not** evaluate) a definite integral which gives the area of the surface obtained by rotating the curve C about y-axis.



$$\Delta SA = 2\pi \underset{\substack{\parallel\\ x}}{r}\,\Delta\ell = 2\pi x\sqrt{(\Delta x)^2+(\Delta y)^2} = 2\pi x\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

$$SA = 2\pi\int_1^5 x\sqrt{1+\frac{1}{4(x-1)}}\,dx$$

OR



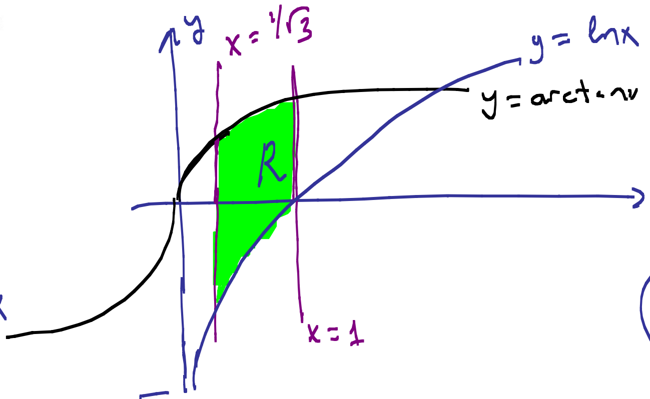
$$\Delta SA = 2\pi x\sqrt{(\Delta x)^2+\Delta y^2} = 2\pi(y^2+1)\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}$$

$$= 2\pi\int_0^2 (y^2+1)\sqrt{1+4y^2}\,dy$$

$y=\sqrt{x-1} \Rightarrow x = y^2+1 \Rightarrow \frac{dx}{dy} = 2y$

Question 6 (10+10 pts) Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \frac{1}{\sqrt{3}}$ and $x = 1$.

a) Find the area of the region R.



$$\text{Area} = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x)\, dx$$

$$= \int_{1/\sqrt{3}}^{1} \arctan x\, dx - \int_{1/\sqrt{3}}^{1} \ln x\, dx$$

$u = 1 + x^2$
$du = 2x\,dx$

$$\int \underbrace{\arctan x}_{u}\, \underbrace{dx}_{dv} = x \arctan x - \int \frac{x}{1+x^2}\, dx$$

$du = \frac{1}{1+x^2}\, dx$

$$= x \arctan x - \frac{1}{2}\ln(1+x^2) + c$$

$v = x$

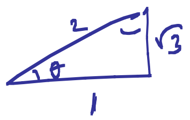
$$\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv} = x \ln x - \int 1\, dx = x\ln x - x + d$$

$du = \frac{1}{x}\, dx$

$v = x$

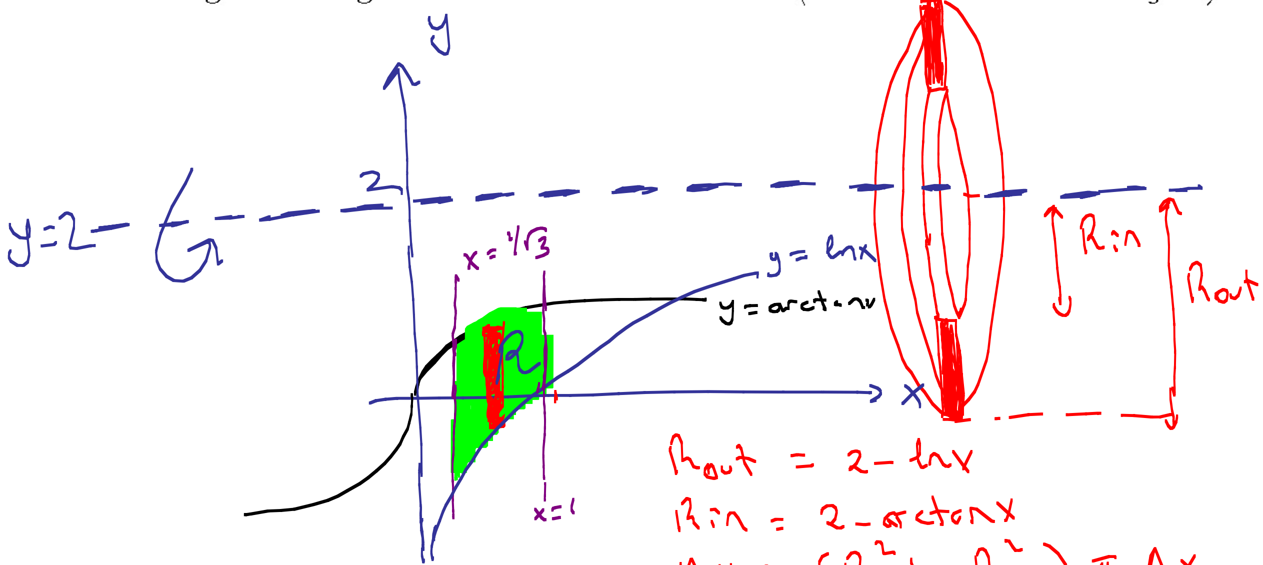
$$= \left[x \arctan x - \frac{1}{2}\ln(1+x^2) - x\ln x + x \right]_{1/\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} + 1\right) - \left(\frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} - \frac{1}{2}\ln\left(1+\frac{1}{3}\right) - \frac{1}{\sqrt{3}}\ln\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)$$



$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} + 1\right) - \left(\frac{1}{\sqrt{3}}\,\frac{\pi}{6} - \frac{1}{2}\ln\left(\frac{4}{3}\right) - \frac{1}{\sqrt{3}}\ln\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)$$

b) A solid S is obtained by rotating the region R in part (a), about $y = 2$ line. Write a definite integral which gives the volume of the solid S. *(Do **not** evaluate the integral.)*



$R_{out} = 2 - \ln x$
$R_{in} = 2 - \arctan x$
$\Delta V = (R_{out}^2 - R_{in}^2)\, \pi\, \Delta x$

$$V = \pi \int_{1/\sqrt{3}}^{1} \left[(2 - \ln x)^2 - (2 - \arctan x)^2 \right] dx$$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY Final Exam	
Code : *Math 119* Acad. Year : *2013-2014* Semester : *Fall* Coordinator : *Muhiddin Uğuz* Date : *January 16, 2014* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	
1 2 3 4 5 6	SHOW YOUR WORK	

Question 1 (6+6=12 pts.) Evaluate the following limits.

a) $\lim_{x\to 0^+} (1+\sin x)^{\cot x}$

$$= e^{\lim_{x\to 0^+} \cot x \cdot \ln(1+\sin x)} = e^{\lim_{x\to 0^+} \frac{\overset{\to 1}{\cos x} \cdot \overset{\to 0}{\ln(1+\sin x)}}{\sin x \to 0}} \quad \left(\frac{0}{0} \text{ type}\right)$$

since e^x is continuous

$$\overset{\text{L'Hopis Rule}}{=} e^{\lim_{x\to 0^+} \frac{-\overset{\to 0}{\sin x} \cdot \overset{\to 0}{\ln(1+\sin x)} + \frac{\cos x}{1+\sin x} \to 1}{\cos x \to 1}} = e^1 = e$$

b) $\lim_{x\to\infty} \dfrac{(\ln x)^2}{\sqrt{x}}$ $\left(\frac{\infty}{\infty} \text{ type}\right)$

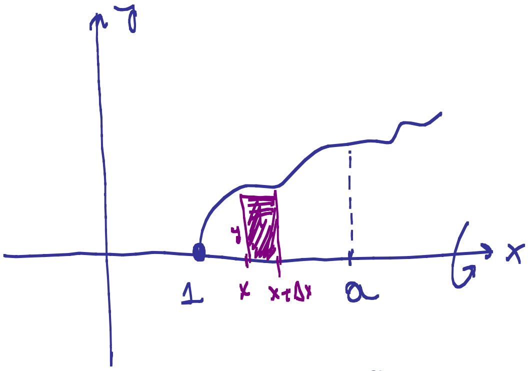
$$\overset{\text{L'Hopis Rule}}{=} \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cdot(\ln x)\frac{1}{x}\cdot 2\sqrt{x} = 4\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \quad \left(\frac{\infty}{\infty} \text{ type}\right)$$

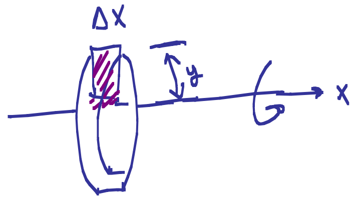
$$\overset{\text{L'Hopis R.}}{=} 4\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 4\lim_{x\to\infty}\frac{1}{x}\,2\sqrt{x} = 8\lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$$

Question 2 (8+6=14 pts) Suppose that $a > 1$ is a constant and consider the curve

$$y = \sqrt{\frac{\ln x}{a^2 x}}\ , \qquad 1 \le x \le a.$$

a) Compute the volume $V(a)$ obtained by rotating the given curve around the x-axis.





$$\Delta V = r^2 \cdot \pi h$$
$$= y^2 \pi \Delta x$$
$$= \frac{\ln x}{a^2 x} \pi \Delta x$$

$$V(a) = \frac{\pi}{a^2} \int_1^a \frac{\ln x}{x}\, dx = \frac{\pi}{a^2} \int_0^{\ln a} u\, du = \frac{\pi}{a^2} \frac{u^2}{2} \Big|_0^{\ln a}$$

$u = \ln x$

$du = \frac{1}{x} dx$

$$= \frac{\pi}{2a^2} \ln^2 a$$

b) Show that $V(a)$ is maximum when $a = e$.

$V(a) = \frac{\pi}{2a^2} \ln^2 a$ is continuous on $[1, \infty)$ and differentiable on $(1, \infty)$

$= \frac{\pi}{2} \left(\frac{\ln a}{a}\right)^2 \Rightarrow V'(a) = \frac{\pi}{\cancel{2}} \cdot \cancel{2} \cdot \frac{\ln a}{a} \cdot \frac{\left(\frac{1}{\cancel{a}} \cdot \cancel{a} - \ln a\right)}{a^2}$

$\Rightarrow V'(a) = \pi \frac{\ln a}{a^3} (1 - \ln a) = 0 \Rightarrow \ln a = 0$ or $\ln a = 1$

$\Rightarrow a = 1$ or $a = e$

a		1		e	
$V'(a)$	////		+	0	−
$V(a)$	///		↗		↘

$V(a)$ is increasing on $[1, e]$ and decreasing on $[e, \infty)$,

Hence its absolute MAX. is at $x = e$

(by first derivative test for absolute max)

Question 3 (6+6+6=18 pts.) Evaluate the following integrals.

a) $\int \tan^5 x \sec x \, dx \quad = \int \tan^4 x \ \sec x \cdot \tan x \, dx = \int (u^4 - 2u^2 + 1)\, du$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$\tan^4 x = \sec^4 x - 2\sec^2 x + 1$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \frac{u^5}{5} - \frac{2}{3}u^3 + u + c = \frac{\sec^5 x}{5} - \frac{2}{3}\sec^3 x + \sec x + C$

(OR $= \int \frac{\sin^5 x}{\cos^6 x}\, dx = \int \frac{\sin^4 x}{\cos^6 x} \sin x \, dx = -\int \frac{(1-u^2)^2}{u^6}\, du = \int -u^{-6} + 2u^{-4} - u^{-2}\, du = ---$)

$u = \cos x$

$du = -\sin x \, dx$

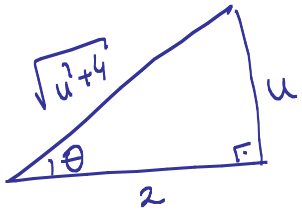
b) $\int \frac{1}{\sqrt{t^2 - 6t + 13}}\, dt \quad = \int \frac{1}{\sqrt{(t-3)^2 + 4}}\, dt = \int \frac{1}{\sqrt{u^2+4}}\, du$

$t^2 - 6t + 13 = t^2 - 6t + 9 - 9 + 13$

$= (t-3)^2 + 4$

$u = t - 3$

$du = dt$



$\tan\theta = \frac{u}{2}$

$\sec^2\theta \, d\theta = \frac{1}{2} du$

$= \int \frac{\cos\theta}{2} \cdot 2\sec^2\theta \, d\theta$

$= \int \cos\theta \frac{1}{\cos^2\theta}\, d\theta = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$

$= \ln\left| \frac{\sqrt{u^2+4}}{2} + \frac{u}{2} \right| + C$

$= \ln\left| \frac{\sqrt{t^2 - 6t + 13}}{2} + \frac{t-3}{2} \right| + c$

c) $\int \frac{3x^3 - 3x^2 + x - 1}{x^2(2x^2+1)}\, dx \quad = \rightarrow$

$$\frac{3x^3 - 3x^2 + x - 1}{x^2(2x^2+1)} = \underset{(2x^3+x)}{\frac{A}{x}} + \underset{(2x^2+1)}{\frac{B}{x^2}} + \underset{(x^2)}{\frac{Cx + D}{2x^2+1}} = \frac{2Ax^3 + Ax + 2Bx^2 + B + Cx^3 + Dx^2}{x^2(2x^2+1)}$$

$\Rightarrow \quad 3x^3 - 3x^2 + x - 1 = (2A + C)x^3 + (2B + D)x^2 + (A)x + B$

$\Rightarrow \quad \boxed{\begin{matrix} B = -1 \\ A = 1 \end{matrix}}$

$2B + D = -2 + D = -3 \rightarrow \boxed{D = -1}$

$2A + C = 2 + C = 3 \Rightarrow \boxed{C = 1}$

$\rightarrow = \int \frac{1}{x} - \frac{1}{x^2} + \frac{x-1}{2x^2+1}\, dx = \ln|x| + \frac{1}{x} + \int \frac{x}{2x^2+1}\, dx - \int \frac{dx}{1 + (\sqrt{2}x)^2}$

$= \ln|x| + \frac{1}{x} + \frac{1}{4}\ln(2x^2+1) - \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x) + C$

Question 4 (6+6+6=18 pts)

Determine whether the following integrals are convergent or divergent.

a) $\int_{2014}^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{c\to\infty} \int_{2014}^{c} \frac{\ln x}{x^2}\,dx = \lim_{c\to\infty} \int_{\ln 2014}^{\ln c} \underbrace{t}_{u}\cdot \underbrace{e^{-t}\,dt}_{dv}$

improper at ∞

$t = \ln x$

$dt = \frac{1}{x}dx$

$e^t = x$

$du = dt$

$v = -e^{-t}$

$= \lim_{c\to\infty} \left[-te^{-t} + \int e^{-t}\,dt\right]_{\ln 2014}^{\ln c} = -\lim_{c\to\infty} \left[\frac{t}{e^t} + e^{-t}\right]_{t=\ln 2014}^{t=\ln c}$

$= -\lim_{c\to\infty} \left[\frac{\ln c}{c} + \frac{1}{c} - \frac{\ln 2014}{2014} - \frac{1}{2014}\right] = \frac{1}{2014}(\ln 2014 + 1) < \infty$

$\frac{\ln c}{c} \to 0$, $\frac{1}{c} \to 0$

$\therefore$ given improper integral is convergent

b) $\int_{1}^{\infty} \frac{1-\cos x}{x^{\frac{5}{2}}+x+119}\,dx$

$= f(x) \geq 0 \;\; \forall x \in [1,\infty]$

since

$0 \leq f(x) \leq \frac{2}{x^{5/2}+x+119} \leq \frac{2}{x^{5/2}} \quad \forall x \in [1,\infty)$

we have

$0 \leq \int_1^\infty f(x)\,dx \leq \underbrace{2\int_1^\infty \frac{1}{x^{5/2}}\,dx}_{\text{convergent by p test } (p=\frac{5}{2}>1)}$

Hence by comparison test,

$\int_1^\infty f(x)\,dx$ is convergent.

c) $\int_{0}^{\infty} \frac{e^x}{x^{\frac{2}{3}}}\,dx$ improper at $x=0$ and $x=\infty$

consider $\int_{A>0}^{\infty} \frac{e^x}{x^{2/3}}\,dx$ improper at $x=\infty$

$f(x) \geq 0 \;\; \forall x \in [1,\infty)$

$\lim_{x\to\infty} \frac{e^x/x^{2/3}}{1/x} = \lim_{x\to\infty} \frac{e^x}{x^{2/3}}\cdot\frac{x}{1} = \lim_{x\to\infty} e^x\cdot x^{1/3} = \infty$

Hence for LARGE value of x, say for $x \geq A$, we have

$\frac{e^x}{x^{2/3}} \gg \frac{1}{x} > 0$. Thus $\int_A^\infty \frac{e^x}{x^{2/3}}\,dx \gg \underbrace{\int_A^\infty \frac{1}{x}\,dx}_{\text{divergent by p test}} > 0$

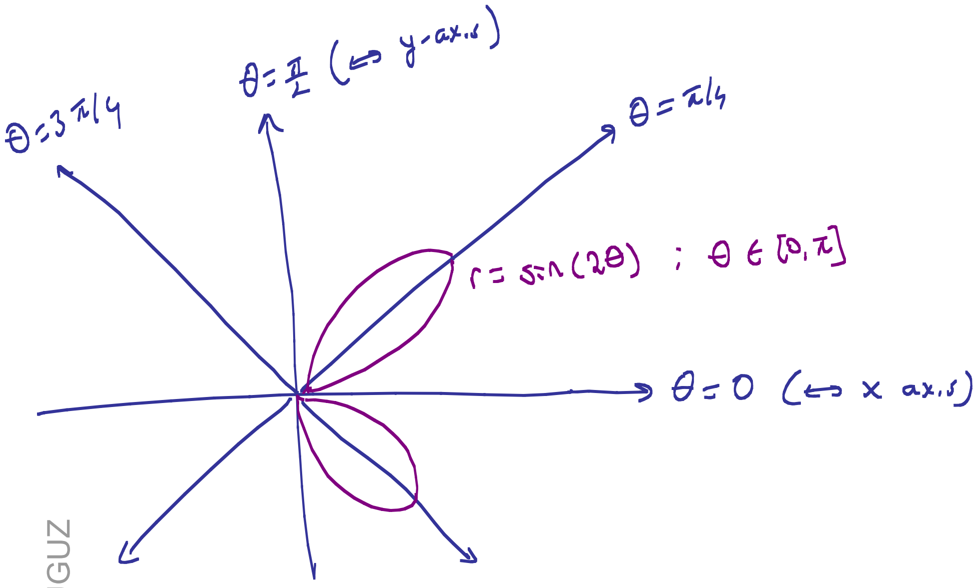
$\Rightarrow \int_A^\infty \frac{e^x}{x^{2/3}}\,dx$ is divergent by limit comp. test $\Rightarrow \int_0^\infty \frac{e^x}{x^{2/3}}\,dx$ is divergent.

Name: ....**Lastname:** ....**Student Id:** .... **Signature:** ....

Question 5 (10+8=18 pts)

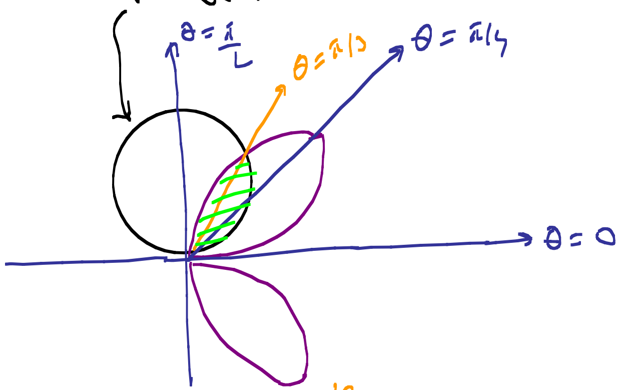
a) Sketch the curve given by the polar equation $r = \sin 2\theta$ for $\theta \in [0, \pi]$.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π
2θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$7\pi/6$	$5\pi/4$	$4\pi/3$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$r = \sin(2\theta)$	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{1}}{2}$	0



b) Find the area of the region that lies inside both the curve $r = \sin 2\theta$ and the circle $r = \sin\theta$ for $\theta \in [0, \pi]$.

$r = \sin\theta \Rightarrow r^2 = r\sin\theta \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \Rightarrow x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$



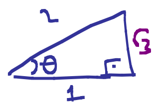
To find intersection points, set

$\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow$ either $\sin\theta = 0 \Rightarrow \theta = 0$

or $1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pi/3$



$\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} \sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} \sin^2 2\theta \, d\theta =$$

$$= \frac{1}{2}\left[\frac{1}{2}\int_0^{\pi/3}(1 - \cos 2\theta)d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(1 - \cos 4\theta)d\theta\right]$$

$$= \frac{1}{4}\left[\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/3} + \left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_{\pi/3}^{\pi/2}\right] = \frac{1}{4}\left[\left(\cancel{\frac{\pi}{3}} - \frac{\sin 2\pi/3}{2}\right) - (0 - 0) + \left(\frac{\pi}{2} - \cancel{\frac{\sin 2\pi}{4}}^{\,0}\right) - \left(\cancel{\frac{\pi}{3}} - \frac{\sin 4\pi/3}{4}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{\sqrt{3}}{4} + \frac{\pi}{2} - \frac{\sqrt{3}}{8}\right] = \frac{\pi/2 - \frac{3\sqrt{3}}{8}}{4} = \frac{4\pi - 3\sqrt{3}}{32}$$

Question 6 (20 pts) Given $y = f(x) = \dfrac{x}{(\ln 3x) - 1}$.

a) Find all vertical and horizontal asymptotes (if there exists any) of the graph of $f(x)$.

$f(x)$ is continuous $\forall x : \ln(3x) - 1 \neq 0$.

- $\ln 3x - 1 = 0 \iff \ln 3x = 1 \iff 3x = e \iff x = e/3$ is a candidate for V.A.

check: $\lim\limits_{x \to e/3^+} \dfrac{x}{\ln(3x) - 1} = +\infty \Rightarrow \boxed{x = e/3 \text{ is V.A}}$ (similarly $\lim\limits_{x \to \frac{e}{3}^-} \dfrac{x}{\ln(3x) - 1} = -\infty$)

since domain of $f(x)$ is $(0, \infty) \setminus \{e/3\}$, we also need to check if $x = 0$ is a V.A.

check $\lim\limits_{x \to 0^+} \dfrac{x}{\ln(3x) - 1} = 0 \neq \infty \Rightarrow x = 0$ is not a V.A.

- $\lim\limits_{x \to +\infty} \dfrac{x}{\ln(3x) - 1} \overset{\text{L'Hopital R.}}{=} \lim\limits_{x \to +\infty} \dfrac{1}{\frac{3}{3x}} = \lim\limits_{x \to +\infty} x = +\infty \Rightarrow \boxed{\text{no H.A.}}$

b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

$f'(x) = \dfrac{\ln(3x) - 1 - x\left(\frac{1}{x}\right)}{(\ln(3x) - 1)^2} = \dfrac{\ln(3x) - 2}{(\ln 3x - 1)^2} = 0 \Rightarrow \ln 3x = 2 \Rightarrow 3x = e^2 \Rightarrow x = \frac{1}{3}e^2$; $(\ln 3x - 1)^2 \geq 0$

$\ln 3x - 1 = 0 \Rightarrow \ln 3x = 1 \Rightarrow x = \frac{1}{3}e \notin \text{Domain}$ } $x = \frac{1}{3}e^2$ is the only critical point.

x	0		$e/3$		$e^2/3$	
f'	//////	$-$	‖	$-$	0	$+$
f	//////	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$		$\nearrow$

f is increasing on $[e^2/3, \infty)$

f is decreasing on $(0, \frac{e}{3})$ and on $(\frac{e}{3}, \frac{e^2}{3}]$

c) Find the intervals on which f is concave up and the intervals on which f is concave down and the point(s) of inflection.

$f''(x) = \left[\frac{1}{x}(\ln(3x) - 1)^{\cancel{2}} - (\ln(3x) - 2)\, 2(\cancel{\ln 3x - 1})\left(\frac{1}{x}\right)\right] / (\ln(3x) - 1)^{\cancel{4}\,3}$

$= \dfrac{(\ln(3x) - 1) - 2\ln(3x) + 4}{x(\ln(3x) - 1)^3} = \dfrac{3 - \ln(3x)}{x(\ln(3x) - 1)^3}$ where $x \neq \frac{1}{3}e$ and $x > 0$

$3 - \ln 3x = 0 \Rightarrow \ln 3x = 3 \iff x = \frac{e^3}{3}$

$\ln 3x - 1 = 0 \iff x = \frac{e}{3}$

x	0		$e/3$		$e^3/3$	
$3 - \ln 3x$	//////	$+$		$+$	0	$-$
$\ln 3x - 1$	//////	$-$	0	$+$		$+$
f''	//////	$-$	‖	$+$		$-$

f is concave up on $(\frac{e}{3}, \frac{e^3}{3})$

f is concave down on $(0, \frac{e}{3})$ and $(\frac{e^3}{3}, \infty)$

$x = e^3/3$ is the inflection point

d) Determine all local extreme values and then classify them as local maximum or local minimum. Domain$(f) = (0, \frac{e}{3}) \cup (\frac{e}{3}, \infty)$ is open so no end points.

The set of all critical points (points in domain such that $f' = 0$ or f' does not exist) is $\{e^2/3\}$ as determined in part (b)

and table in part (b) shows that (by First Derivative Test) $x = \frac{e^2}{3}$ is the only local minimum point ($f(\frac{e^2}{3}) = \frac{e^2}{3}$ local min value). No local MAX exists.

M E T U Department of Mathematics

Math 119 Calculus with Analytic Geometry Exam I 06.04.2013		
Last Name: Name : Student No:	Instructor : Time : 13: 30 Duration : 100 *minutes*	Signature
5 QUESTIONS ON 4 PAGES		TOTAL 90 POINTS

1	2	3	4	5			

(8+8+8 pts) **1.** Evaluate the following limits or explain why it does not exist.

(a) $\lim\limits_{x\to -2} \dfrac{|x+2|}{|x|-2}$

$$\lim_{x\to -2^-} \frac{|x+2|}{|x|-2} = \lim_{x\to -2^-} \frac{-(x+2)}{-x-2} = +1$$

$$\lim_{x\to -2^+} \frac{|x+2|}{|x|-2} = \lim_{x\to -2^+} \frac{x+2}{-x-2} = -1$$

$$\Rightarrow \lim_{x\to -2^-} \frac{|x+2|}{|x|-2} \neq \lim_{x\to -2^+} \frac{|x+2|}{|x|-2}$$

$\Rightarrow$ limit does not exist. $\square$

(b) $\lim\limits_{x\to 1} \dfrac{\tan x - x}{x^3}$

at $x=1$ $(\tan x - x)$ is defined and nonzero, and $x^3 \neq 0$

$$\Rightarrow \lim_{x\to 1} \frac{\tan x - x}{x^3} = \tan(1) - 1 \quad \square$$

(c) $\lim\limits_{x\to -\infty} \ln(\sqrt{x^2+1}+x)$

$$\lim_{x\to -\infty} \ln\left(\sqrt{x^2+1}+x\right) = \lim_{x\to -\infty} \ln\left((\sqrt{x^2+1}+x)\cdot \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}\right) = \lim_{x\to -\infty} \ln\left(\frac{1}{\sqrt{x^2+1}-x}\right)$$

As $x\to -\infty$, $(\sqrt{x^2+1}-x) \to \infty \Rightarrow \left(\dfrac{1}{\sqrt{x^2+1}-x}\right) \to 0^+$

Hence, $\lim\limits_{x\to -\infty} \ln\left(\dfrac{1}{\sqrt{x^2+1}-x}\right) = -\infty \quad \square$

(8+8+8 pts) 2.

a) Find y' if $y = \tan^2\left(\frac{x+1}{x-1}\right)$.

Using the chain rule,

$$y' = 2\tan\left(\frac{x+1}{x-1}\right) . \sec^2\left(\frac{x+1}{x-1}\right) . \left(\frac{x-1-(x+1)}{(x-1)^2}\right)$$

$$\Rightarrow \quad y' = \frac{-4}{(x-1)^2} . \tan\left(\frac{x+1}{x-1}\right) . \sec^2\left(\frac{x+1}{x-1}\right) \quad \square$$

b) Find y' if $y = 2^{x^2-3x+8}$

Taking the logarithm of both sides;

$$\ln y = \ln 2^{x^2-3x+8} \quad \Rightarrow \quad \ln y = (x^2-3x+8) . \ln 2$$

Then, differentiating both sides;

$$\frac{y'}{y} = (2x-3)\ln 2 \quad \Rightarrow \quad y' = (2x-3) . (\ln 2) . 2^{x^2-3x+8} \quad \square$$

c) Let $F(x) = f(3 + 2\sqrt{x})$. If $f'(7) = 1$, find $F'(4)$.

Using the chain rule;

$$F'(x) = f'(3+2\sqrt{x}) . \left(\frac{1}{\sqrt{x}}\right)$$

Then;

$$F'(4) = f'(3+2.2) . \frac{1}{2}$$

$$\Rightarrow \quad F'(4) = f'(7) . \frac{1}{2}$$

$$(f'(7)=1) \Rightarrow \quad F'(4) = \frac{1}{2} \quad \square$$

(9+9 pts) **3.** Let

$$f(x) = \begin{cases} \frac{\sin(2x^2)+x^3}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

(a) Show that the function f is continuous at $x = 0$.

existence of limit

We should check that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

continuity

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(2x^2)+x^3}{x} = \lim_{x\to 0^-}\left(\frac{\sin(2x^2)}{x} + x^2\right)$$

$$= \lim_{x\to 0^-}\left(\underbrace{2x}_{\downarrow 0}\cdot \underbrace{\frac{\sin(2x^2)}{(2x^2)}}_{\downarrow 1} + \underbrace{x^2}_{\downarrow 0}\right) = 0 \quad \text{(Since each limit exists)}$$

Similarly, $\lim_{x\to 0^+} f(x) = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$ (i.e. limit exists)

Moreover, $f(0) = 0 \Rightarrow f$ is continuous at $x = 0$ □

(b) Find $f'(0)$ if it exists.

If exists, $f'(0) = \lim_{x\to 0} \frac{f(x) - \overset{0}{f(0)}}{x - 0} = \lim_{x\to 0} \frac{\sin(2x^2)+x^3}{x^2}$

So,

$$\lim_{x\to 0} \frac{\sin(2x^2)+x^3}{x^2} = \lim_{x\to 0}\left(2\underbrace{\frac{\sin(2x^2)}{2x^2}}_{\downarrow 1} + \underbrace{x}_{\downarrow 0}\right) = 2 \quad \text{(Since each limit exists)}$$

Hence, $f'(0) = 2$ □

(12 pts) **4.** Show that the equation $x^3 + x^2 + 2x + 5 = 0$ has a root, but no more than one.

Let $f(x) = x^3 + x^2 + 2x + 5$, then $f(x)$ is continuous and differentiable on the whole real line.

Note that; $\left.\begin{array}{l} f(0) = 5 > 0 \\ f(-2) = -3 < 0 \end{array}\right\} \Rightarrow$ thus, by Intermediate Value theorem (f is continuous on $[-2,0]$) there is a point $c_1 \in (-2,0)$ such that $\underline{f(c_1) = 0}$ (i.e. $x = c_1$ is a root of $f(x) = 0$).

Assume that there is another point $c_2 \neq c_1$ such that $\underline{f(c_2) = 0}$ (i.e. assume that $x = c_2$ is another root of $f(x) = 0$).

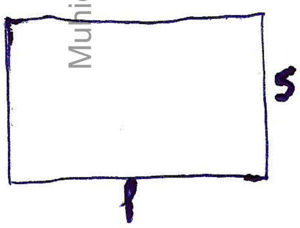
Applying the Mean Value theorem on $[c_1, c_2]$ (f is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2))

$$\Rightarrow \frac{\overbrace{f(c_2)}^{0} - \overbrace{f(c_1)}^{0}}{c_2 - c_1} = f'(d) \Rightarrow f'(d) = 0 \text{ for some } d \in (c_1, c_2)$$

But, $f'(x) = 3x^2 + 2x + 2 \neq 0$ ($\Delta = 4 - 24 = -20 < 0$), hence there is no such d

$\Rightarrow x = c_1$ is the only root $\square$

(12 pts) **5.** If the short side of a rectangle is increasing at a rate of 3 cm per second and the long side is decreasing at a rate of 2 cm per second. Determine how fast the area of the rectangle is changing when the short side is 4 cm and the long side is 6 cm.



The area function of a rectangle is; (a function of time t)

$$\boxed{A(t) = s(t) \cdot l(t)}$$

The short side is increasing at a rate of 3cm/sec $\Rightarrow \boxed{\frac{ds}{dt} = +3}$

The long side is decreasing at a rate of 2cm/sec. $\Rightarrow \boxed{\frac{dl}{dt} = -2}$

Derivating $A(t)$:

$$\frac{dA}{dt} = \frac{ds}{dt} \cdot l(t) + s(t) \cdot \frac{dl}{dt}$$

$\underline{\text{When } s = 4,\ l = 6} \rightarrow \frac{dA}{dt} = 3 \cdot 6 + 4 \cdot (-2)$

$\Rightarrow \frac{dA}{dt} = 18 - 8 = 10 \Rightarrow$ the area is $\underline{\text{increasing}}$ at a rate of $10\,cm^2/sec$ when $s = 4cm$, $l = 6cm$

$\square$

METU Department of Mathematics

Math 119 Calculus with Analytic Geometry		Exam II 04.05.2013	
Last Name : Name : Student No:	Instructor : Time : 13: 30 Duration : 100 *minutes*	Signature	
5 QUESTIONS ON 4 PAGES		TOTAL 90 POINTS	
1 | 2 | 3 | 4 | 5			

SHOW YOUR WORK!

(8+8+8 pts) 1. Evaluate the followings limits:

(a) $\lim_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$

$$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}\ \underset{L'H.}{\overset{\left[\frac{0}{0}\right]}{=}}\ \lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}=\lim_{x\to 0}\frac{-\cos^2 x}{\cos x+1}=-\frac{1}{2}\ \square$$

$$\left[1-\sec^2 x=\frac{\cos^2 x-1}{\cos^2 x}\right]$$

(b) $\lim_{x\to 0^+}(1+\sin x)^{\frac{1}{x}}$

$$\lim_{x\to 0^+}(1+\sin x)^{\frac{1}{x}}=\lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+\sin x)}\ \underset{\downarrow}{=}\ e^{\lim_{x\to 0^+}\frac{\ln(1+\sin x)}{x}}=e^{1}=e\ \square$$

[exponential function is continuous]

$$\lim_{x\to 0^+}\frac{\ln(1+\sin x)}{x}\ \underset{L'H'}{\overset{\left[\frac{0}{0}\right]}{=}}\ \lim_{x\to 0^+}\frac{\cos x}{1+\sin x}=1$$

(c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin(\frac{i\pi}{n})$.

Let $f(x)=\sin(x)$, then, on $[a,b]$, $\int_a^b f(x)\,dx \overset{\text{[Riemann Sum]}}{=} \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\, f(c_i)$

where $\Delta x=\dfrac{b-a}{n}$

$c_i\in[x_{i-1},x_i]$, $i=1,2,\dots,n$

$x_0=a$, $x_n=b$

$x_i=a+\frac{b-a}{n}\cdot i$

$i=1,2,\dots,n$.

Thus, we have that;

$\Delta x=\frac{\pi}{n}\ \Rightarrow\ b-a=\pi$

$c_i=\frac{i\pi}{n}=\frac{b-a}{n}i \xrightarrow{n\to\infty} 0\ \Rightarrow\ \boxed{c_i=x_i}$ with $a=0$, $b=\pi$ $(c_n=\pi)$

$i=1,\dots,n.$

Hence;

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{i\pi}{n}\right)=\int_0^{\pi}\sin(x)\,dx=-\cos x\Big|_0^{\pi}=-(\underset{-1}{\cos\pi}-\underset{1}{\cos 0})$$

$$=2\ \square$$

$$\left[\,Dom(f) = \mathbb{R} \setminus \{\pm 2\}\,\right]$$

(3+3+3+3+4 pts) **2.** Let $f(x) = \dfrac{x^2-5}{x^2-4}$.

a) Find the asymptotes of $y = f(x)$.

$\lim\limits_{x\to\pm\infty} \dfrac{x^2-5}{x^2-4} = 1 \Downarrow$ $\boxed{y=1}$ is horizontal asymptote

$\lim\limits_{x\to -2^-} \dfrac{x^2-5}{x^2-4} = -\infty$, $\lim\limits_{x\to -2^+} \dfrac{x^2-5}{x^2-4} = +\infty \Downarrow$ $\boxed{x=-2}$ is vertical asymptote

$\lim\limits_{x\to 2^-} \dfrac{x^2-5}{x^2-4} = +\infty$, $\lim\limits_{x\to 2^+} \dfrac{x^2-5}{x^2-4} = -\infty \Downarrow$ $\boxed{x=2}$ is vertical asymptote.

b) Find the intervals where f is increasing and the intervals where f is decreasing.

$$f'(x) = \frac{2x(x^2-4) - 2x(x^2-5)}{(x^2-4)^2} = \frac{2x}{(x^2-4)^2} \Rightarrow \begin{array}{l} f'(x) > 0 \text{ if } x>0 \\ f'(x) < 0 \text{ if } x<0 \end{array}$$

$\Rightarrow$ f is increasing on $(0,2)$ and $(2,\infty)$
f is decreasing on $(-\infty,-2)$ and $(-2,0)$

c) Find the intervals where f is concave up and the intervals where f is concave down.

$$f''(x) = \frac{2.(x^2-4)^2 - 2x.2(x^2-4).2x}{(x^2-4)^4} = \frac{2(x^2-4) - 8x^2}{(x^2-4)^3} = \frac{-(6x^2+8)}{(x^2-4)^3}$$

$\Rightarrow f''(x) > 0$ if $-2<x<2$
$f''(x) < 0$ if $x<-2$, $x>2$

f is concave up on $(-2,2)$
f is concave down on $(-\infty,-2)$ and $(2,\infty)$

d) Find the local extreme values of $f(x)$.

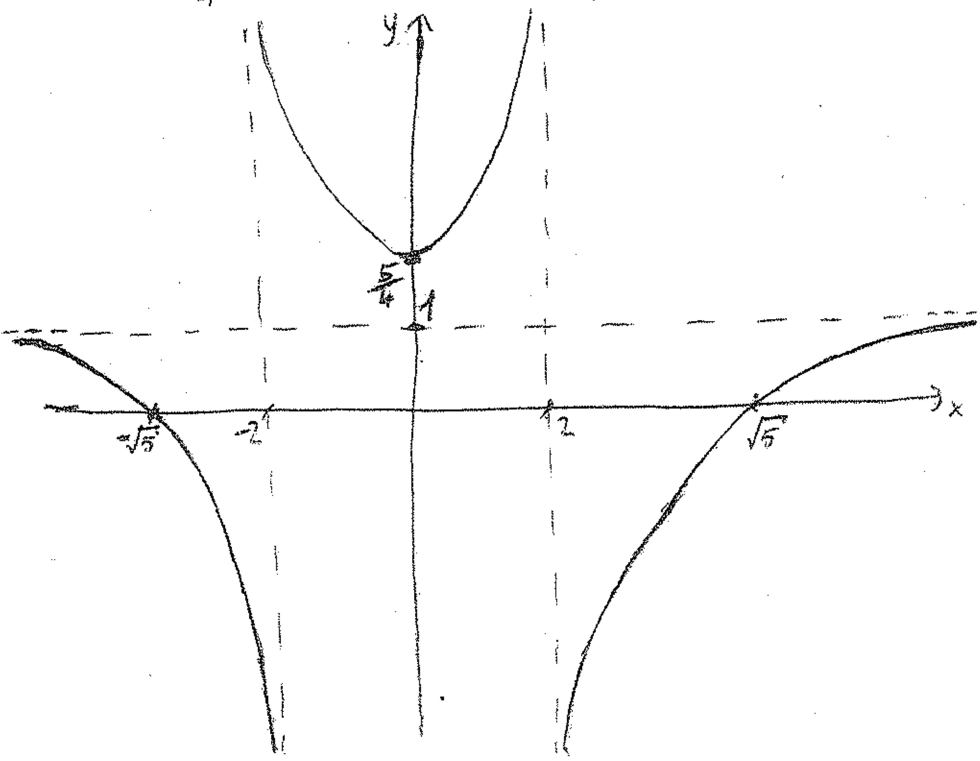
$f'(x) = \dfrac{2x}{(x^2-4)^2} = 0 \iff \boxed{x=0} \to$ critical point
(No singular point, $x=\pm 2 \notin Dom(f)$)

f increases on the right of $x=0$ and decreases on the left of $x=0$
$\Rightarrow f(0) = \frac{5}{4}$ is a local minimum value.

e) Sketch the graph of $f(x)$.

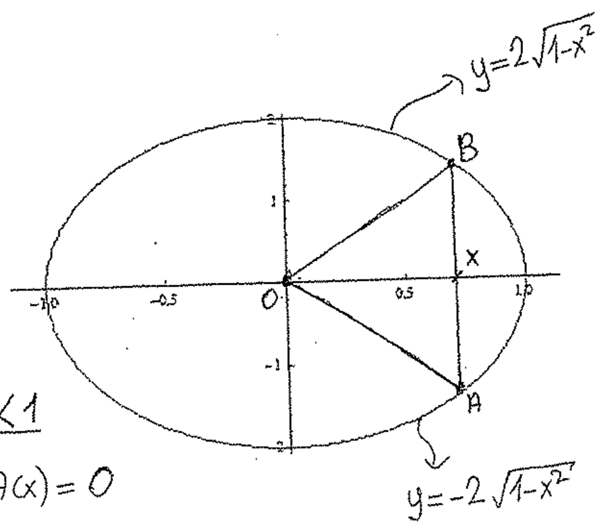
x		-2		0		2	
f'	$-$		$-$		$+$		$+$
f''	$-$		$+$		$+$		$-$

$-\infty \,||\, +\infty$ local minimum $+\infty \,||\, -\infty$



(15 pts) 3. Find the area of the largest triangle with the following properties: one vertex is $(0,0)$, the other vertices are on the ellipse $4x^2+y^2=4$ and one side is parallel to the y-axis.

$4x^2+y^2 \Rightarrow y=\pm 2\sqrt{1-x^2}$



Consider the triangle AOB, with its height is x unit and its base AB is $4\sqrt{1-x^2}$ unit. Then, the area function is;

$$A(x)=\frac{1}{4}\cdot 4\sqrt{1-x^2}\,x \Rightarrow A(x)=2x\sqrt{1-x^2}$$

where $0<x<1$

On $(0,1)$, $A(x)>0$ and $\lim_{x\to 0^+} A(x)=\lim_{x\to 1^-} A(x)=0$

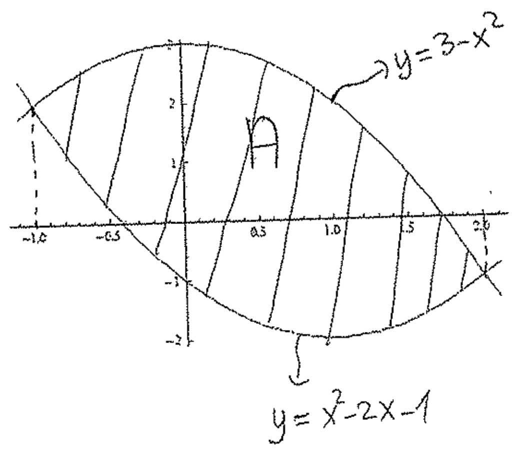
$$A'(x)=2\sqrt{1-x^2}+2x\cdot\frac{-2x}{2\sqrt{1-x^2}} \Rightarrow A'(x)=\frac{2(1-2x^2)}{\sqrt{1-x^2}}$$

$A'(x)=0 \underset{(x>0)}{\Leftrightarrow} \boxed{x=\frac{\sqrt{2}}{2}}$ critical point

Thus, $A\left(\frac{\sqrt{2}}{2}\right)=1$ is the largest area for such a triangle.

(15 pts) 4. Sketch the region between the parabolas $y=3-x^2$, $y=x^2-2x-1$ and find the area of this region.

$3-x^2=x^2-2x-1 \Rightarrow x^2-x-2=0$
$\Leftrightarrow x=-1 , x=2$



On the interval $[-1,2]$, the value of the curve $y=3-x^2$ is greater than the value of the curve $y=x^2-2x-1$. Then,

$$A=\int_{-1}^{2}\left[3-x^2-(x^2-2x-1)\right]dx = \int_{-1}^{2}(-2x^2+2x+4)\,dx$$

$$=\left(-\frac{2x^3}{3}+x^2+4x\right)\Big|_{-1}^{2}$$

$$=-\frac{16}{3}+4+8-\left(\frac{2}{3}+1-4\right)=9 \ \square$$

(10+10 pts) 5. Evaluate the following integrals:

a) $\int x \arctan(x^2)\,dx$.

Let $u = \arctan(x^2)$ $\quad dv = x\,dx$

$du = \frac{2x}{1+x^4}dx$ $\quad v = \frac{x^2}{2}$

[Integration by parts: $\int u\,dv = uv - \int v\,du$]

Then; $\int x \arctan(x^2)\,dx = \frac{x^2}{2}\arctan(x^2) - \int \frac{x^3}{1+x^4}dx$

[Method of substitution

$u = 1+x^4$

$du = 4x^3dx$

$\Rightarrow x^3dx = \frac{1}{4}du$

$\Rightarrow \int \frac{x^3}{1+x^4}dx = \frac{1}{4}\int \frac{1}{u}du$]

$$\Rightarrow \int x\arctan(x^2)\,dx = \frac{x^2}{2}\arctan(x^2) - \frac{1}{4}\ln(1+x^4) + C \quad \square$$

b) $\int \frac{2x\,dx}{x^2+2x+3}$.

$$\int \frac{2x\,dx}{x^2+2x+3} = \int \frac{2(x+1)-2}{(x+1)^2+2}dx = 2\underbrace{\int \frac{x+1}{(x+1)^2+2}dx}_{} - 2\int \frac{1}{2+(x+1)^2}dx$$

$u = (x+1)^2+2$

$du = 2(x+1)dx$

$$\Rightarrow \int \frac{2x\,dx}{x^2+2x+3} = \int \frac{du}{u} - \int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}dx$$

$$\Rightarrow \int \frac{2x\,dx}{x^2+2x+3} = \ln\left((x+1)^2+2\right) - \sqrt{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C \quad \square$$

M E T U Department of Mathematics

Math 119 Calculus with Analytic Geometry Final Exam 30.05.2013		
Last Name: Name : Student No:	Instructor : Time : 13:30 Duration : 120 *minutes*	Signature
6 QUESTIONS ON 6 PAGES		TOTAL 120 POINTS

1	2	3	4	5	6			

SHOW YOUR WORK!

(15 pts) **1.** Find an equation of the tangent line to the curve

$$\frac{x}{y} = \cos(xy\pi)$$

at the point $(-1, 1)$.

$$y\cos(xy\pi) = x$$

Differentiating implicitely;

$$y'.\cos(xy\pi) + y.(-\sin(xy\pi)).(y\pi + y'x\pi) = 1$$

At $(-1,1) \rightarrow$ $y'\cos(-\pi) - \underbrace{\cancel{\sin(-\pi)}}_{0}.(\pi - \pi y') = 1$

$$\Rightarrow \boxed{y'\big|_{(-1,1)} = -1}$$

Thus, the tangent line equation is given by, at $(-1,1)$,

$$y = y'\big|_{(-1,1)}(x-(-1)) + 1$$

$$\Rightarrow \boxed{y = -x}$$

(8+8+8 pts) **2.** Evaluate the followings limits:

(a) $\lim\limits_{x\to\infty} \dfrac{2^x + 3^x}{3^x + 4^x}$

$$= \lim_{x\to\infty} \frac{\cancel{4^x}\left(\left(\frac{2}{4}\right)^x + \left(\frac{3}{4}\right)^x\right)}{\cancel{4^x}\left(\left(\frac{3}{4}\right)^x + 1\right)} = 0$$

$\left(\lim\limits_{x\to\infty} r^x = 0 \quad \text{for} \quad |r|<1\right)$

(b) $\lim\limits_{x\to 0} \dfrac{1}{x^3}\displaystyle\int_0^x \tan(t^2)\,dt$

$$\lim_{x\to 0} \frac{\int_0^x \tan(t^2)\,dt}{x^3} \overset{\left[\frac{0}{0}\right]}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\tan(x^2)}{3x^2} = \lim_{x\to 0} \underbrace{\frac{\sin(x^2)}{x^2}}_{\to 1} \cdot \underbrace{\frac{1}{3\cos(x^2)}}_{\to \frac{1}{3}} = \frac{1}{3}$$

(each limit exists)

(c) $\lim\limits_{x\to\infty} \dfrac{3x + |\sin(2x)|}{x^2}$

$$\lim_{x\to\infty}\left[\underbrace{\frac{3}{x}}_{\to 0} + \frac{|\sin(2x)|}{x^2}\right] = 0$$

Note that, $0 \leq |\sin(2x)| \leq 1$

$$\Rightarrow \underbrace{0}_{\to 0} \leq \underbrace{\frac{|\sin(2x)|}{x^2}}_{\to 0 \text{ as } x\to\infty} \leq \underbrace{\frac{1}{x^2}}_{\to 0 \text{ as } x\to\infty}$$

(by Squeeze Thm.)

(7+7+7 pts) 3. Evaluate the following integrals:

(a) $\int \frac{x+1}{x^2(x^2+1)}\,dx$ (Use partial fraction)

$$\frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \Rightarrow x+1 = \underbrace{(A+C)}_{=0}x^3 + \underbrace{(B+D)}_{=0}x^2 + Ax + B$$

$$\Rightarrow \boxed{A=1},\ \boxed{B=1},\ \boxed{C=-1},\ \boxed{D=-1}$$

Thus; $\int \frac{x+1}{x^2(x^2+1)}dx = \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{x+1}{x^2+1}dx$

$$= \ln|x| - \frac{1}{x} - \underbrace{\int \frac{x}{x^2+1}dx}_{\left[\begin{array}{l} u=x^2+1 \\ du=2x\,dx \end{array}\right]} - \int \frac{dx}{x^2+1} = \ln|x| - \frac{1}{x} - \frac{1}{2}\ln(x^2+1) - \operatorname{Arctan}(x) + C$$

(b) $\int \frac{x^2+x+1}{\sqrt{x}}\,dx$

$$\left[\begin{array}{l} u=\sqrt{x} \\ du=\frac{1}{2\sqrt{x}}dx \end{array}\right] \Rightarrow 2\int (u^4+u^2+1)\,du = 2\left(\frac{x^{5/2}}{5} + \frac{x^{3/2}}{3} + \sqrt{x}\right) + C$$

(c) $\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$ $= \int_0^{\frac{\pi}{6}} u\,du = \left.\frac{u^2}{2}\right|_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}$

$$\left[\begin{array}{l} u=\arcsin x \\ du=\frac{dx}{\sqrt{1-x^2}} \\ x=0 \Rightarrow u=\arcsin(0)=0 \\ x=\frac{1}{2} \Rightarrow u=\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6} \end{array}\right]$$

(10+10 pts) 4. Determine whether the following improper integrals are convergent or divergent.

(a) $\int_0^3 \frac{x\,dx}{(9-x^2)^{3/2}}$ $\rightarrow$ Improper at $x=3$.

$$= \lim_{R\to 3^-} \int_0^R \frac{x\,dx}{(9-x^2)^{3/2}} \quad \boxed{\begin{matrix} u = 9-x^2 \\ du = -2x\,dx \end{matrix}} \quad = \lim_{R\to 3^-} -\frac{1}{2}\int_{x=0}^{x=R} \frac{du}{u^{3/2}}$$

$$= \lim_{R\to 3^-} -\frac{1}{2}\cdot(-2)\cdot\frac{1}{\sqrt{u}}\Big|_{x=0}^{x=R} = \lim_{R\to 3^-} \frac{1}{\sqrt{9-x^2}}\Big|_0^R$$

$$= \lim_{R\to 3^-}\Big(\underbrace{\frac{1}{\sqrt{9-R^2}}}_{\infty} - \frac{1}{3}\Big) = \infty \Rightarrow \text{divergent}$$

(b) $\int_0^\infty \frac{dx}{\sqrt{x}+x^3}$ $\rightarrow$ improper at both $x=0$ and $x\to\infty$

$$= \int_0^1 \frac{dx}{\sqrt{x}+x^3} + \int_1^\infty \frac{dx}{\sqrt{x}+x^3}$$ $\rightarrow$ both terms are convergent, then the given integral is convergent.

(*) $\frac{1}{\sqrt{x}+x^3} < \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent $\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}+x^3}$ converges (by comparison test)

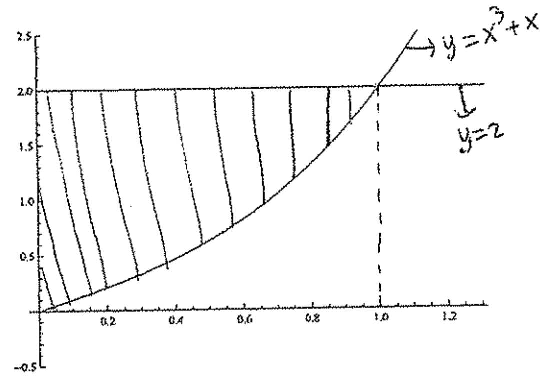
p-integral: $\int_0^a \frac{dx}{x^p}$ converges for $p<1$, $a>0$

(*) $\frac{1}{\sqrt{x}+x^3} < \frac{1}{x^3}$ and $\int_1^\infty \frac{dx}{x^3}$ is convergent $\Rightarrow \int_1^\infty \frac{dx}{\sqrt{x}+x^3}$ converges (by comparison test)

p-integral: $\int_a^\infty \frac{dx}{x^p}$ converges for $p>1$, $a>0$

(2+9+9 pts) **5.** Let R be the plane region bounded by the graph of $y = x^3 + x$ and the lines $x = 0$ and $y = 2$.

(a) Sketch the region R.

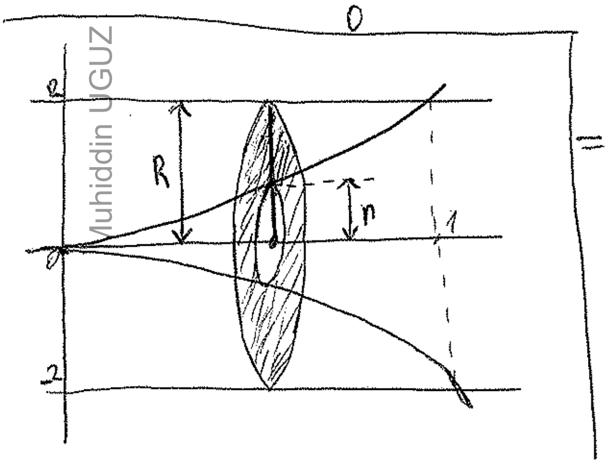


(b) Find the volume of the solid obtained by rotating R about the x–axis.

Disk Method:

$R = 2 \longrightarrow$ radius of outer disk
$r = x^3 + x \rightarrow$ radius of inner disk

$$V = \pi \int_0^1 (R^2 - r^2)\,dx = \pi \int_0^1 \left(4 - (x^3+x)^2\right) dx$$



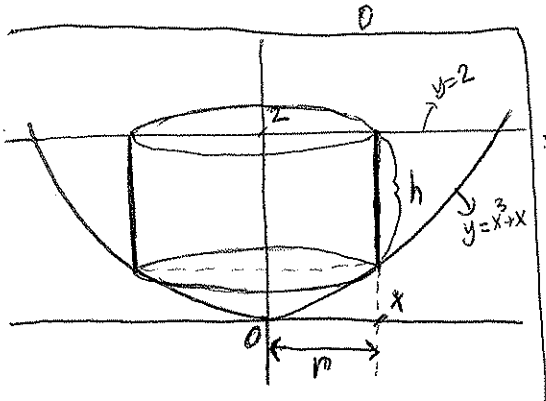
$$= \pi\left(4x - \frac{x^7}{7} - \frac{2x^5}{5} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{328\pi}{105}$$

(c) Find the volume of the solid obtained by rotating R about the y–axis.

Cylindrical Shell Method:

$r = x \longrightarrow$ radius of the cylinder (base radius)
$h = 2 - (x^3 + x) \rightarrow$ height of the cylinder

$$V = 2\pi \int_0^1 rh\,dx = 2\pi \int_0^1 x\left(2 - x^3 - x\right) dx$$



$$= 2\pi\left(x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{14\pi}{15}$$

(10+10 pts) **6.** Write an integral (DO NOT EVALUATE) giving
(a) the arc length of the curve $x^2 + 4y^2 = 9$, $0 \le x \le 1$, $y \ge 0$.

$$\text{arclength} \rightarrow \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

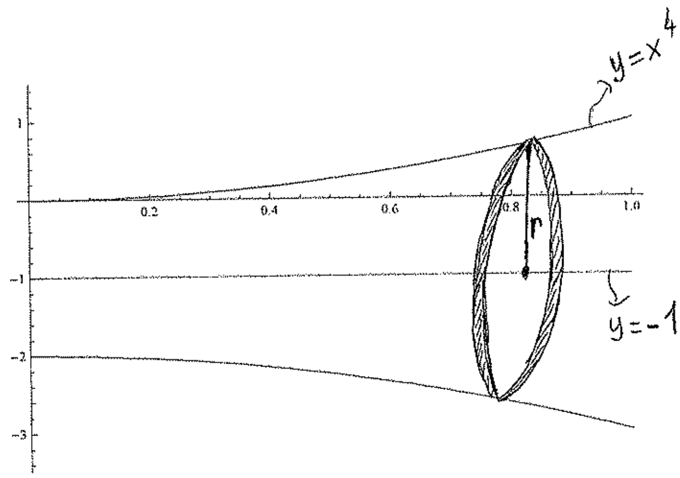
$$x^2+4y^2=9 \Rightarrow 2x+8y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{4y}$$; $\frac{dy}{dx}$ is continuous for $0 \le x \le 1$ and $4y > 0$ there.

So; $$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{16y^2}} = \sqrt{1+\frac{x^2}{36-4x^2}}$$

$$\Rightarrow \text{arclength} \rightarrow \int_0^1 \sqrt{1+\frac{x^2}{36-4x^2}}\,dx$$

(b) the surface area of the surface of revolution obtained by rotating $y = x^4$, $0 \le x \le 1$, about the line $y = -1$.

$$S = 2\pi \int_0^1 \underset{|y|}{r}\,ds$$



$$ds = \sqrt{1+\left(\frac{dy}{dx^2}\right)^2}\,dx \Rightarrow ds = \sqrt{1+16x^6}\,dx$$

$$r = |y| = x^4-(-1) = x^4+1$$

$$\Rightarrow S = 2\pi \int_0^1 (x^4+1)\sqrt{1+16x^6}\,dx$$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	List No.

Code : *Math 119* Acad. Year : *2012-2013* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *November.24.2012* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (7+7+7+7+7 pts) Evaluate the limit or explain why it does not exist:

a) $\lim_{x\to 0}\frac{f(x)-5}{\sin x}$ if $f(0)=5$ and $f'(0)=2$.

$$= \lim_{x\to 0}\frac{f(x)-f(0)}{\sin x} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\,\frac{x}{\sin x} \overset{\text{since each limit exists}}{=} \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\cdot\lim_{x\to 0}\frac{x}{\sin x}$$

$$= f'(0)\cdot 1 = 2$$

b) $\lim_{x\to\infty}\frac{\ln(1+\sin^2 x)}{\arctan x + e^x}$

- $0\le \sin^2 x\le 1 \Rightarrow 1\le 1+\sin^2 x\le 2 \Rightarrow 0\le \ln(1+\sin^2 x)\le \ln 2 \quad \forall x$.
- $\arctan x + e^x > 0$ for large x.

Therefore $0 \le \frac{\ln(1+\sin^2 x)}{\arctan x + e^x} \le \frac{\ln 2}{\arctan x + e^x}$, by squeeze Thm, we have

as $x\to\infty$: 0

$$\lim_{x\to\infty}\frac{\ln(1+\sin^2 x)}{\arctan x + e^x} = 0$$

c) $\lim_{x\to\infty}(e^x - x^2)$

$$= \lim_{x\to\infty} \underbrace{e^x}_{\to\infty}\underbrace{\left(1-\frac{x^2}{e^x}\right)}_{\to 1}.$$

since $\lim_{x\to\infty}\frac{x^2}{e^x}$ ($\frac{\infty}{\infty}$ type) $\overset{\text{L'Hopital Rule}}{=} \lim_{x\to\infty}\frac{2x}{e^x}$ ($\frac{\infty}{\infty}$ type) $\overset{\text{L'Hopital Rule}}{=} \lim_{x\to\infty}\frac{2}{e^x} = 0$

$$= \infty$$

d) $\lim_{x\to\infty}\frac{x+\sqrt[4]{16x^6+x^5-3x+7}}{5x+8-\sqrt{x^3+x-2}}$

$$= \lim_{x\to\infty}\frac{\cancel{x^{3/2}}\left(\frac{1}{x^{1/2}}+\sqrt[4]{16+\frac{1}{x}-\frac{3}{x^5}+\frac{7}{x^6}}\right)}{\cancel{x^{3/2}}\left(\frac{5}{x^{1/2}}+\frac{8}{x^{3/2}}-\sqrt{1+\frac{1}{x^2}-\frac{2}{x^3}}\right)}$$

$$= \frac{\sqrt[4]{16}}{-1} = -2$$

e) $\lim_{x\to 0^+}(1+4x+x^2)^{2/x}$

$$= e^{\lim_{x\to 0^+}\frac{2\ln(1+4x+x^2)}{x}} \quad \left(\tfrac{0}{0}\text{ type}\right)$$

$$\overset{\text{L'Hopital Rule}}{=} e^{\lim_{x\to 0^+}\frac{2(4+2x)}{1+4x+x^2}} = e^8$$

Question 2 (4+6+1 pts) Consider the curve given by the equation

$$xy^2 + \sin((x-1)y) = 4.$$

a) Find all points $(1, y)$ on this curve.

$x = 1 \Rightarrow y^2 + \sin 0 = 4 \Rightarrow y^2 = 4 \Rightarrow y = \mp 2$

$(1,-2)$, $(1,+2)$.

b) Find the equations of the tangent lines to the curve at the points you found in part(a).

Equation of tangent line at (x_0, y_0) on the curve is

$$y = y'\Big|_{(x_0,y_0)} (x - x_0) + y_0$$

using implicit differentiation, we obtain:

$$y^2 + 2xyy' + \cos((x-1)y)\left[y + (x-1)y'\right] = 0$$

when $x_0 = 1$, we have ;

$$y^2 + 2yy' + y = 0$$

when $(x_0, y_0) = (1,-2)$ we have $4 - 4y' - 2 = 0 \Rightarrow y'\Big|_{(1,-2)} = \frac{1}{2}$

when $(x_0, y_0) = (1, 2)$ " " $4 + 4y' + 2 = 0 \Rightarrow y'\Big|_{(1,2)} = -3/2$

Thus

- Tangent line at $(1,-2)$: $y = \frac{1}{2}(x-1) - 2$ ie, $y = \frac{1}{2}x - \frac{5}{2}$
- " " " $(1,2)$: $y = \frac{-3}{2}(x-1) + 2$ ie, $y = \frac{-3}{2}x + \frac{7}{2}$

c) Are any of the tangent lines you found in part(b) parallel? Explain!

Since $\frac{1}{2} \neq \frac{-3}{2}$, that is, since slopes of the lines we obtained are not the same, these lines are not parallel.

Question 3 (7 pts) Let $f(x) = |x^3|$. Find $f'(0)$ by using the definition of derivative and find an equation of the tangent line to the graph of $f(x)$ at the point where $x = 0$.

$$\lim_{x\to 0} \frac{f(x) - \overset{0}{\overset{\shortparallel}{f(0)}}}{x - 0} = \lim_{x\to 0} \frac{|x^3| - 0}{x} = \lim_{x\to 0} \frac{|x^3|}{x} \overset{*}{=} 0$$

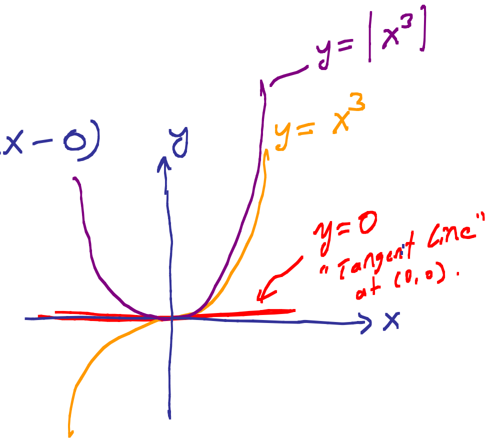
$$*: \left(\begin{array}{l} \lim_{x\to 0^-} \frac{|x^3|}{x} = \lim_{x\to 0^-} \frac{-x^3}{x} = \lim_{x\to 0^-} -x^2 = 0 \\ \lim_{x\to 0^+} \frac{|x^3|}{x} = \lim_{x\to 0^+} \frac{x^3}{x} = \lim_{x\to 0^+} x^2 = 0 \end{array} \right) \text{ or } \left(\lim_{x\to 0} \frac{|x^3|}{x} = \lim_{x\to 0} \frac{x^2|x|}{x} = \lim_{x\to 0} x|x| = 0 \right)$$

Hence $f'(0) = 0$

Equation of tangent line is:

$$y - \underbrace{f(0)}_{\shortparallel\, 0} = \underbrace{f'(0)}_{\shortparallel\, 0}(x - 0)$$

$$\underline{\underline{y = 0}}$$



Question 4 (7 pts) Prove that $\ln x < x - 1$ for all $x > 1$.

given any $x > 1$,

Since $f(t) = \ln t$ is continuous on $[1, x]$ and differentiable on $(1, x)$ $\quad (f'(t) = \frac{1}{t})$

by Mean Value Theorem, we have:

$$\frac{f(x) - f(1)}{x - 1} = f'(c) \quad \text{for some } c \in (1, x)$$

that is

$$\frac{\ln x - \overset{0}{\overset{\shortparallel}{\ln 1}}}{x - 1} = \frac{1}{c} \quad \text{for some } c \in (1, x) \quad (\Rightarrow \tfrac{1}{c} < 1)$$

Hence

$$\frac{\ln x}{x - 1} < 1 \Rightarrow \ln x < x - 1 \quad \forall x > 1$$

or/

$$\boxed{\begin{array}{l} \text{Let } g(t) = \ln t - t + 1 \text{. Then } g(1) = 0 \text{ and } g'(t) = \frac{1}{t} - 1 \\ \text{Since } x > 1 \text{ we have } 1 > \frac{1}{x} \text{ and hence } g'(x) = \frac{1}{x} - 1 < 0 \\ \text{Thus } g(t) \text{ is decreasing and hence } g(x) < g(1) = 0 \ \forall x > 1. \\ \therefore g(x) < 0, \text{ giving } \ln x - x + 1 < 0 \ \forall x > 1 \end{array}}$$

Question 5 (7 pts) Show that $f(x) = x^7 + \arctan x + e^{x-1} - \frac{\pi}{4}$ has an inverse function (defined on the range of $f(x)$) and find the derivative of the inverse function $f^{-1}(x)$ at the point $x_0 = f(1)$.

Since $f'(x) = \underbrace{7x^6}_{\geq 0} + \underbrace{\frac{1}{1+x^2}}_{\neq 0} + \underbrace{e^{x-1}}_{\neq 0} \cancel{\geq} 0 \quad \forall x$, we know that

f is (strictly) increasing $\forall x$, and hence one to one function.

Thus f has inverse defined on the range of f.

since $f^{-1}(f(x)) = x \quad \forall x \in Dom(f)$, taking derivative of both sides, and using chain rule, we obtain;

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

$$\therefore (f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad (\cancel{\geq} 0)$$

Hence $$(f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{1}{7 + \frac{1}{2} + 1} = \frac{1}{\frac{17}{2}} = \frac{2}{17}$$

Question 6 (7 pts) Use the definition of limit to show that $\lim_{x \to 1}(2x + 119) = 121$.

given any $\varepsilon > 0$,

we want to find $\delta > 0$ such that for all x with $0 < |x-1| < \delta$, we have $|2x + 119 - 121| < \varepsilon$

Note that $|2x+119-121| = 2|x-1| < 2\delta$ for $0 < |x-1| < \delta$

choose $\delta = \frac{\varepsilon}{2} > 0$.

Then, for all x with $0 < |x-1| < \delta$

we have

$$|2x+119-121| = 2|x-1| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim_{x \to 1} (2x + 119) = 121$

First Name: .... **Last Name:** .... **Id:** .... **Section** :.......

Question 7 (6+6+6 pts) For the given function $f(x)$, find the derivative function $f'(x)$. DO NOT SIMPLIFY:

a) $f(x) = \cos^2\sqrt{x} + \dfrac{1}{\ln x}$

$$f'(x) = 2(\cos\sqrt{x})(-\sin\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + \frac{-1/x}{\ln^2 x}$$

b) $f(x) = \dfrac{\arcsin\left(\frac{1}{1+x^2}\right)}{1+\cos x}$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \Rightarrow (\arcsin g(x))' = \frac{1}{\sqrt{1-g^2(x)}} \cdot g'(x)$$

$$= \frac{\arcsin(g(x))}{1+\cos x}$$

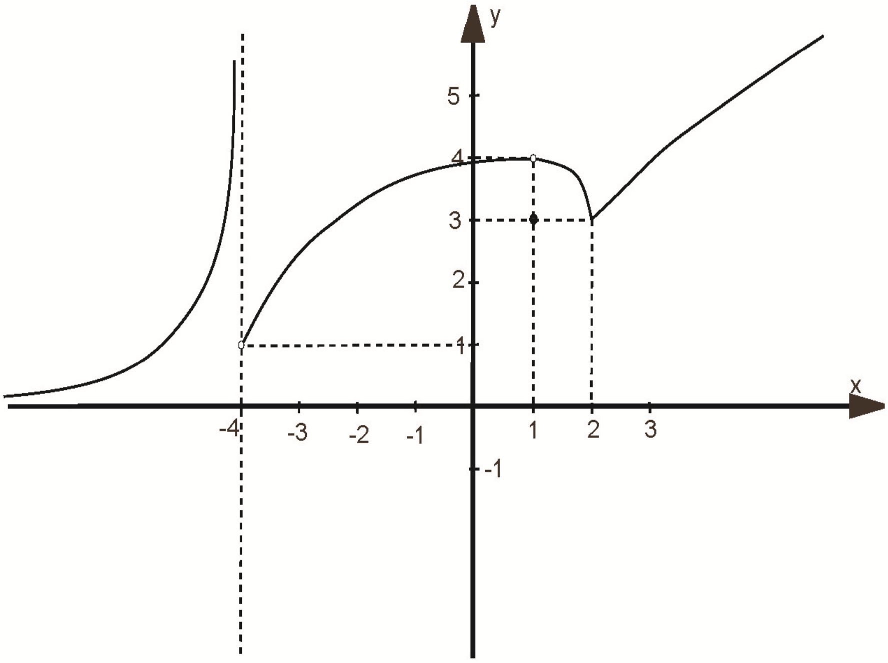
$$f'(x) = \frac{\frac{1}{\sqrt{1-\frac{1}{(1+x^2)^2}}}\left(\frac{-2x}{(1+x^2)^2}\right) \cdot (1+\cos x) - \arcsin\left(\frac{1}{1+x^2}\right)(-\sin x)}{(1+\cos x)^2}$$

c) $f(x) = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x = e^{x\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)}$

$$\left(e^{g(x)}\right)' = g'(x)e^{g(x)}$$

$$f'(x) = \left[\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right) + x\,\frac{1}{1+\frac{3}{x}+\frac{5}{x^2}} \cdot \left(\frac{-3}{x^2} + \frac{-10}{x^3}\right)\right]\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$$

Question 8 (2+2+2+2 pts) The graph of the function $f(x)$ is given below. Answer the following questions:



(a) Find the limit $\lim_{x \to -4^+} f(x)$ if it exists.

$$\lim_{x \to -4^+} f(x) = 1$$

(b) Find the limit $\lim_{x \to 1} f(x)$ if it exists.

$$\lim_{x \to 1} f(x) = 4$$

(c) Find all intervals on which $f(x)$ is continuous.

$(-\infty, -4)$, $(-4, 1)$, $(1, \infty)$

(d) Find all intervals on which $f(x)$ is differentiable.

$(-\infty, -4)$, $(-4, 1)$, $(1, 2)$, $(2, \infty)$

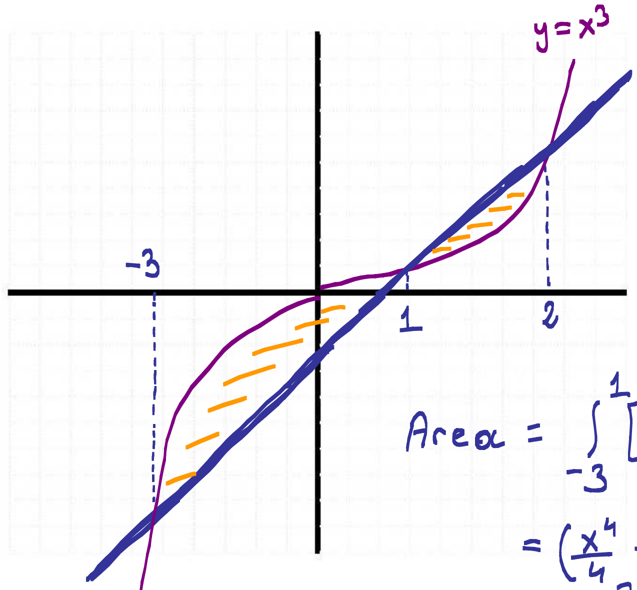
M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 2	List No.

Code : *Math 119* Acad. Year : *2012-2013* Semester : *Fall* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *December.22.2012* Time : *9:30* Duration : *100 minutes*	5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	SHOW YOUR WORK	

Question 1 (18 pts) Sketch and find the area of the region bounded by $y = x^3$ and $y = 7x - 6$. (Hint: One of the points of intersection is (1,1).)



$y = 7x - 6$: $x = 0 \Rightarrow y = -6$
$y = 0 \Rightarrow x = \frac{6}{7}$

To find intersection points, set:

$x^3 = 7x - 6 \Rightarrow x^3 - 7x + 6 = 0$

x=1 is given as one of the root.

$\Rightarrow x^3 - 7x + 6 = (x-1)(x^2 + ax + b)$

$= (x-1)(x+3)(x-2)$

$\Rightarrow x = -3, 1, 2$

$$\text{Area} = \int_{-3}^{1} [x^3 - (7x-6)]\,dx + \int_{1}^{2} [(7x-6) - x^3]\,dx$$

$$= \left(\frac{x^4}{4} - \frac{7x^2}{2} + 6x\right)\Big|_{-3}^{1} + \left(\frac{7x^2}{2} - 6x - \frac{x^4}{4}\right)\Big|_{1}^{2}$$

$$\text{Area} = \left[\left(\frac{1}{4} - \frac{7}{2} + 6\right) - \left(\frac{81}{4} - \frac{63}{2} - 18\right)\right] + \left[\left(\frac{28}{2} - 12 - \frac{16}{4}\right) - \left(\frac{7}{2} - 6 - \frac{1}{4}\right)\right]$$

$$= \left(\frac{1}{4} - \frac{81}{4} - \frac{16}{4} + \frac{1}{4}\right) + \left(\frac{-7}{2} + \frac{63}{2} + \frac{28}{2} - \frac{7}{2}\right) + (+6 + 18 - 12 + 6) = \frac{-95}{4} + \frac{77}{2} + 18$$

$$= \frac{-95 + 154 + 72}{4} = \frac{131}{4} \text{ unit square}$$

Question 2 (10 pts) Find the limit $\lim_{n\to\infty}\left(\frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \cdots + \sqrt{1+\frac{n-1}{n}} + \sqrt{1+\frac{n}{n}}\right)\right) = \rightarrow$

by interpreting it as a definite integral.

$$\rightarrow = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{i}{n}} \overset{*}{=} \int_0^1 \sqrt{1+x}\,dx$$

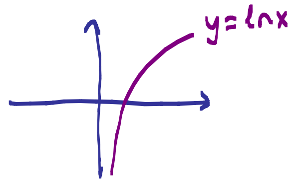
(*): Consider • the partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ of the interval $[0,1]$ into n-equal length subintervals. ($\Rightarrow \Delta x_i = \frac{1-0}{n} = \frac{1}{n} \Rightarrow x_i = \frac{i}{n}$)
• function $f(x) = \sqrt{1+x}$.
Then given limit is limit of right sum (or, since f is increasing, upper sum) for f with respect to $\mathcal{P}_n$; and hence equal to

$$\int_0^1 \sqrt{1+x}\,dx = \int_1^2 u^{1/2}\,du = \frac{2}{3} u^{3/2}\Big|_1^2 = \frac{2}{3}(2\sqrt{2} - 1)$$

$u = 1 + x$
$du = dx$

(or, by similar argument, given limit is equal to $\int_1^2 \sqrt{x}\,dx$, or, ...)

Question 3 (4+4+8+8+8 pts) Let $f(x) = \dfrac{x + \ln x}{x}$, $x > 0$.



a) Find the vertical asymptotes of the graph of f if there are any.

$f(x)$ is continuous on $(0,\infty)$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \underbrace{\frac{1}{x}}_{\to +\infty} \underbrace{(x+\ln x)}_{\to -\infty} = -\infty$$

so $x=0$ is the only vertical asymptote.

b) Find the horizontal asymptotes of the graph of f if there are any.

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x+\ln x}{x} \quad \left(\tfrac{\infty}{\infty} \text{ typ}\right)$$

L'Hopi Rule $\overset{}{=} \lim_{x\to+\infty} 1 + \frac{1}{x} = 1 \Rightarrow y = 1$ is the only horizontal asymptote.

c) Find the intervals where f is increasing, where f is decreasing. Determine the local minimum and maximum points of f if there are any.

$$f'(x) = \frac{1-\ln x}{x^2} = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$ is the only critical point.

x		e	
$f'(x)$	+	0	−
$f(x)$	↗		↘

$f(x)$ is increasing on $(0, e]$

decreasing on $[e, \infty)$

and by first derivative test, $f(e) = \frac{1}{e} + 1$ is local maximum.

d) Find the intervals where the graph of f is concave up, where it is concave down. Find the inflection points of the graph of of f if there are any.

$$f''(x) = \frac{-x - 2x(1-\ln x)}{x^4} = -\frac{3-2\ln x}{x^3} = 0 \Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$$

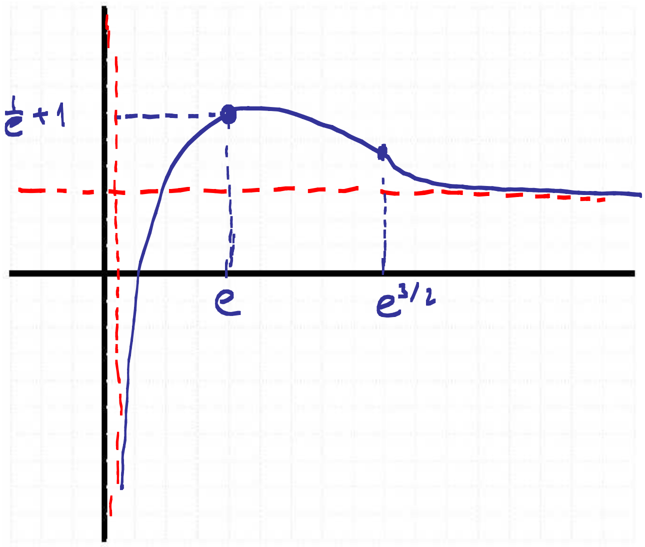
x		$e^{3/2}$	
$f''(x)$	−	0	+
$f(x)$	∩		∪

f is concave up on $(e^{3/2}, \infty)$

concave down on $(0, e^{3/2})$

$P = \left(e^{3/2}, \frac{3}{2e^{3/2}} + 1\right)$ is inflection point on the graph

e) Sketch the graph of f.



Question 4 (16 pts) A cylinder with an open top has total surface area $27\pi \text{cm}^2$. Find the radius of such a cylinder which has maximum volume. Explain why your answer gives the maximum volume.

$27\pi = 2\pi r h + \pi r^2 \Rightarrow h = \frac{27\pi - \pi r^2}{2\pi r} = \frac{27 - r^2}{2r}$

Volume $V = \pi r^2 h$

$\Rightarrow V(r) = \pi r^2 \left(\frac{27 - r^2}{2r}\right) = \frac{\pi}{2} r (27 - r^2)$

$= \frac{\pi}{2}(27r - r^3) \quad ; \quad 0 \le r \le \sqrt{27}$

if $r = 0$, or $r = \sqrt{27}$, $V = 0$

$V(r)$ is continuous on closed interval $[0, \sqrt{27}]$, so it has maximum value.

$\frac{dV}{dr} = \frac{\pi}{2}(27 - 3r^2) = 0 \Rightarrow r^2 = 9 \Rightarrow r = 3$ is only critical point.

$V(3) = \frac{\pi}{2}(81 - 27) > 0$ so $r = 3$ gives MAX. Volume.

Question 5 (8+8+8 pts) Evaluate:

a) $\int \underbrace{x^2}_{u} \underbrace{e^x\,dx}_{dv} = x^2e^x - 2\int \underbrace{x}_{u}\, \underbrace{e^x\,dx}_{dv} = x^2e^x - 2\left(xe^x - \int e^x\,dx\right)$

$du = 2x\,dx$ $\quad v = e^x$ $\qquad\qquad du = dx$ $\quad v = e^x$

$= x^2e^x - 2xe^x + 2e^x + C$

$= e^x(x^2 - 2x + 2) + C$

b) $\int \frac{1+x}{1+\sqrt{x}}\,dx = \int \frac{1+u^2}{1+u}\cdot 2u\,du = 2\int \frac{u^3+u}{1+u}\,du$

Let $x = u^2$. $(\sqrt{x} = u > 0)$

then $dx = 2u\,du$

$$\begin{array}{r|l} u^3+u & u+1 \\ \hline -\;u^3+u^2 & u^2-u+2 \\ -u^2+u & \\ -\;-u^2-u & \\ 2u & \\ 2u+2 & \\ -2 & \end{array}$$

$= 2\int \left(u^2 - u + 2 - \frac{2}{u+1}\right) du$

$= 2\left(\frac{u^3}{3} - \frac{u^2}{2} + 2u - 2\ln|u+1|\right) + C$

$= 2\left[\frac{x^{3/2}}{3} - \frac{x}{2} + 2\sqrt{x} - 2\ln(\sqrt{x}+1)\right] + C$

c) $\int \frac{8x+1}{x^2(x^2+4)}\,dx$

$\frac{8x+1}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} = \frac{x^3(A+C) + x^2(B+D) + x(4A) + 4B}{x^2(x^2+4)}$

(x^3+4x) $\quad (x^2+4)$ $\quad (x^2)$

$4B = 1 \Rightarrow \boxed{B = 1/4}$ $\qquad A + C = 0 \Rightarrow \boxed{C = -2}$

$4A = 8 \Rightarrow \boxed{A = 2}$ $\qquad B + D = 0 \Rightarrow \boxed{D = -1/4}$

Hence

$\int \frac{8x+1}{x^2(x^2+4)}\,dx = 2\int \frac{1}{x}\,dx + \frac{1}{4}\int \frac{1}{x^2}\,dx + \int \frac{-2x - \frac{1}{4}}{x^2+4}\,dx$

$= 2\ln|x| - \frac{1}{4x} - \int \frac{2x}{x^2+4}\,dx - \frac{1}{4}\int \frac{1}{4+x^2}\,dx$

$= 2\ln|x| - \frac{1}{4x} - \ln(x^2+4) - \frac{1}{16}\int \frac{1}{1+(\frac{x}{2})^2}\,dx$

$= 2\ln|x| - \frac{1}{4x} - \ln(x^2+4) - \frac{1}{8}\arctan\left(\frac{x}{2}\right) + C$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY Final Exam	List No.

Code : *Math 119* Acad. Year : *2012-2013* Semester : *Fall* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *January.10.2013* Time : *9:30* Duration : *119 minutes*	8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (6+6 pts)

(a) Find $\dfrac{d}{dx}\left(x^{\sin(x^2)}\right)$ $= \dfrac{d}{dx} e^{\sin(x^2) \cdot \ln x}$

$$= e^{\sin(x^2).\ln x}\left[2x\cos(x^2).\ln x + \frac{\sin(x^2)}{x}\right]$$

$$= x^{\sin(x^2)}\left[2x\cos(x^2).\ln x + \frac{\sin(x^2)}{x}\right]$$

(b) Find $\dfrac{dy}{dx}$ at the point $(x, y) = (1, 0)$ if $e^{xy} = e^y \ln(x^2) + y^2 + 1$.

$$\frac{d}{dx}(e^{xy}) = \frac{d}{dx}\left(e^y \ln(x^2) + y^2 + 1\right)$$

$$e^{xy}(y + xy') = y'e^y \ln(x^2) + e^y \frac{2}{x} + 2yy'$$

substitude $x=1$, $y=0$ to get

$$y' = 2$$

Question 2 (6+6 pts) Evaluate the limit if it exists or explain why it does not exist.
DO NOT USE L'HOPITAL RULES!

(a) $\lim_{x\to-\infty}\left(\sqrt{x^{10/3}-5x^2}-\sqrt{x^{10/3}+5x^2}\right) = \lim_{x\to-\infty} \dfrac{\cancel{x^{10/3}}-5x^2-\cancel{x^{10/3}}-5x^2}{\sqrt{x^{10/3}-5x^2}+\sqrt{x^{10/3}+5x^2}}$

$$= \lim_{x\to-\infty} \frac{-10x^2}{-x^{5/3}\left(\sqrt{1-\frac{5}{x^{4/3}}}+\sqrt{1+\frac{5}{x^{4/3}}}\right)}$$

$$= 10 \lim_{x\to-\infty} \frac{x^{1/3} \to -\infty}{\sqrt{1-\frac{5}{x^{4/3}}}+\sqrt{1+\frac{5}{x^{4/3}}} \to 2} = -\infty$$

(b) $\lim_{x\to-1} \dfrac{2-\sqrt{f(x)}}{x+1}$ if $f(-1)=4$, $f(x)>0$ for all x and $f'(-1)=\pi$.

$$\lim_{x\to-1} \frac{4-f(x)}{(2+\sqrt{f(x)})(x+1)} = -\lim_{x\to-1}\left(\frac{f(x)-4}{x+1}\cdot\frac{1}{2+\sqrt{f(x)}}\right)$$

since each limit exists

$$= -\underbrace{\lim_{x\to-1}\frac{f(x)-f(-1)}{x-(-1)}}_{= f'(-1)=\pi} \cdot \underbrace{\lim_{x\to-1}\frac{1}{2+\sqrt{f(x)}}}_{=\frac{1}{2+\sqrt{f(-1)}}=\frac{1}{4}}$$

since $f'(-1)$ exists, $f(x)$ is continuous at -1 and hence $\lim_{x\to-1} f(x) = f(-1)$

$$= -\frac{\pi}{4}$$

Question 3 (6 pts) Find the equations of the tangent lines to the curve $y=x^3+x$ which pass through the point $(2,2)$.

Take a point (a, a^3+a) on the curve $y=x^3+x$.

slope of line through (a, a^3+a) and $(2,2)$ is $m = \dfrac{2-a^3-a}{2-a}$

slope of tangent line is $\dfrac{dy}{dx} = 3x^2+1$

at $x=a$, slope $= 3a^2+1$

$$\frac{2-a^3-a}{2-a} = 3a^2+1 \Rightarrow 2-a^3-a = 6a^2+2-3a^3-a$$

$$\Rightarrow 2a^3-6a^2=0 \Rightarrow 2a^2(a-3)=0$$

$$\Rightarrow a_1=0, \quad a_2=3$$

at $(0,0)$, slope is 1 and hence equation of tangent line: $y=x$

at $(3,30)$, slope is 28 " " " " " $y=28(x-3)+30$

Question 4 (6+6 pts) Find

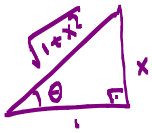
(a) $\displaystyle\int \frac{x^3\,dx}{\sqrt{1+x^2}} = \int (u^2-1)\,du = \frac{u^3}{3} - u + C = \frac{(1+x^2)^{3/2}}{3} - \sqrt{1+x^2} + C$

Let $u = \sqrt{1+x^2}$

then $du = \dfrac{x}{\sqrt{1+x^2}}\,dx$

$x^2 = u^2 - 1$

OR



$\left.\begin{array}{l} \tan\theta = x \\ \cos\theta = \dfrac{1}{\sqrt{1+x^2}} \\ \sec^2\theta\,d\theta = dx \end{array}\right\}$

$\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \tan^3\theta \cdot \cos\theta \sec^2\theta\,d\theta$

$\displaystyle = \int \frac{\sin^3\theta}{\cos^3\theta}\cos\theta\,\frac{1}{\cos^2\theta}\,d\theta$

$\displaystyle = \int \frac{(1-\cos^2\theta)}{\cos^4\theta}\sin\theta\,d\theta$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$\displaystyle = -\int \frac{1-u^2}{u^4}\,du = \int u^{-2} - u^{-4}\,du$

$\displaystyle = -\frac{1}{\cos\theta} + \frac{1}{3\cos^3\theta} + C = -\sqrt{1+x^2} + \frac{(1+x^2)^{3/2}}{3} + C$

(b) $\displaystyle\int \frac{dx}{x(1+x^{119})} = \int \frac{1}{119(u-1)u}\,du \Rightarrow$

Let $u = 1 + x^{119}$

then $du = 119x^{118}\,dx$

hence $dx = \dfrac{du}{119x^{118}} \Rightarrow \dfrac{dx}{x} = \dfrac{du}{119x^{119}} = \dfrac{du}{119(u-1)}$

$\dfrac{1}{(u-1)u} = \dfrac{A}{u-1} + \dfrac{B}{u} = \dfrac{(A+B)u - B}{(u-1)u}$

$\left.\begin{array}{l} A+B = 0 \\ -B = 1 \end{array}\right\}$ $\begin{array}{l} A = 1 \\ B = -1 \end{array}$

$\displaystyle \Rightarrow = \frac{1}{119}\int\left(\frac{1}{u-1} - \frac{1}{u}\right)du = \frac{1}{119}\Big[\ln|u-1| - \ln|u|\Big] + C$

$\displaystyle = \frac{1}{119}\ln\left|\frac{x^{119}}{1+x^{119}}\right| + C$

Question 5 (8+8 pts)

(a) Evaluate the integral $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^4}$ if it is convergent.

$$\int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{c\to\infty}\int_0^c \frac{x\,dx}{1+x^4} = \lim_{c\to\infty} \frac{1}{2}\arctan(x^2)\Big|_0^c$$

$$= \frac{1}{2}\lim_{c\to\infty} \arctan(c^2) - \cancelto{0}{\arctan(0)}$$

$$= \frac{1}{2}\,\frac{\pi}{2} = \frac{\pi}{4}$$

(b) Determine whether the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^7+x^{1/3}}$ is convergent or divergent.

improper at 0 & at ∞

consider

$$I_1 = \int_0^1 \frac{dx}{x^7+x^{1/3}} \quad \text{and} \quad I_2 = \int_1^\infty \frac{dx}{x^7+x^{1/3}}dx$$

(if both I_1 & I_2 are convergent, then so is given integral,
if one of I_1 or I_2 is divergent, " " " " ")

I_1: since $x>0$, $x^7+x^{1/3} > x^{1/3}$ and hence $0 < \dfrac{1}{x^7+x^{1/3}} < \dfrac{1}{x^{1/3}}$

moreover since $\displaystyle\int_0^1 \frac{1}{x^{1/3}}dx = \lim_{c\to 0^+} \frac{3}{2}(1-c^{2/3}) = 3/2$ convergent,
(or simply "convergent by p test")

By comparison test
I_1 is also convergent

or/ $\displaystyle\lim_{x\to 0} \frac{\frac{1}{x^7+x^{1/3}}}{\frac{1}{x^{1/3}}} = \lim_{x\to 0}\frac{x^{1/3}}{x^{1/3}(x^{20/3}+1)} = 1$, a nonzero number, hence
by Limit comparison test, $\displaystyle\int_0^1 \frac{dx}{x^7+x^{1/3}}$ converges if and only if $\displaystyle\int_0^1 \frac{1}{x^{1/3}}dx$ conv.
since 2^{nd} one is convergent by p-test, so is I_1.

I_2: since $x \geq 1$, $x^7+x^{1/3} > x^7 > 0$, Hence $0 < \dfrac{1}{x^7+x^{1/3}} < \dfrac{1}{x^7}$

$\displaystyle\int_1^\infty \frac{1}{x^7}dx = \lim_{c\to\infty}\int_1^c \frac{1}{x^7}dx = \frac{1}{6}$ conv. Hence by comparison test I_2 is convergent

Hence $I = I_1 + I_2$ is convergent.

NAME AND STUDENT ID: SIGNATURE:

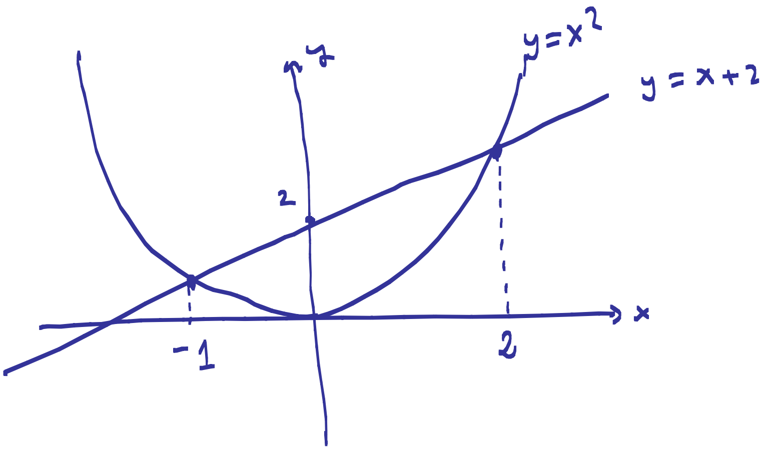
Question 6 (3+8+8 pts) Let R be the finite region bounded by the curves $y = x^2$ and $y = x + 2$.

(a) Sketch the region R.

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

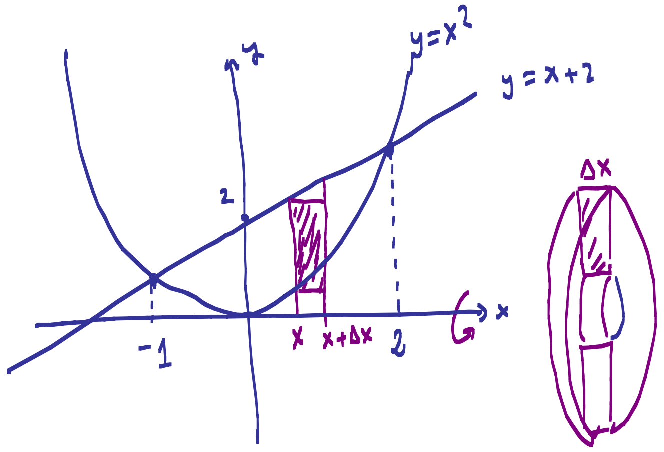
$(x-2)(x+1) = 0$

$x = +2, \quad x = -1$



(b) Set up a definite integral which gives the **volume** of the solid obtained by rotating the region R around the x-**axis**. DO NOT EVALUATE THE INTEGRAL!

slicing (disc) method.

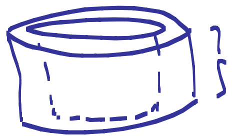


$\Delta V = \pi (R^2 - r^2)\,\Delta x$ where $R = x+2$, $r = x^2$

Hence

$$V = \pi \int_{-1}^{2} \left[(x+2)^2 - x^4\right] dx$$

(c) Set up a definite integral which gives the **volume** of the solid obtained by rotating the region R around the **line** $x = -1$. DO NOT EVALUATE THE INTEGRAL!



$(x+2) - x^2$

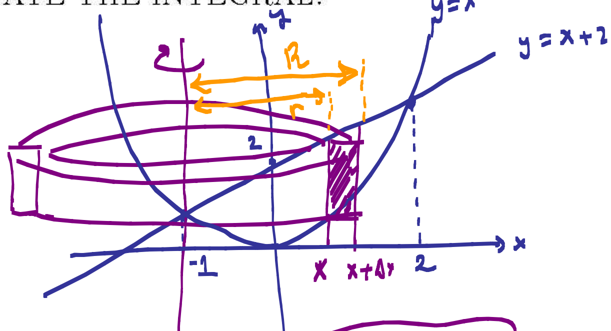
shell mtd.

$\Delta V = 2\pi r\,\Delta x\,\left[(x+2) - x^2\right]$

where $r = 1 + x$

Hence

$$V = \lim_{\|\Delta x\| \to 0} \sum \Delta V = 2\pi \int_{-1}^{2} (1+x)(x+2-x^2)\,dx$$



OR

$\Delta V = \pi (R^2 - r^2)(x+2-x^2)$

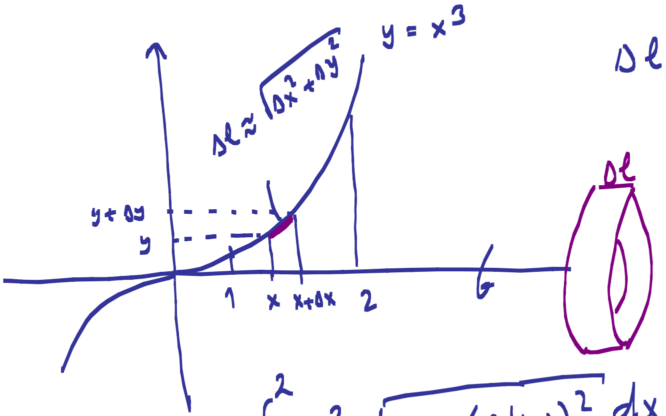
where $R = 1 + x + \Delta x$

$r = 1 + x$

$\Rightarrow R^2 - r^2 = 2(1+x)\Delta x + (\Delta x)^2$

Hence $V = 2\pi \int_{-1}^{2} (1+x)(x+2-x^2)\,dx$

Question 7 (8 pts) Find the **area of the surface** obtained by rotating the curve $y = x^3$, $1 \le x \le 2$, around the x-**axis**.



$$\Delta \ell \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$\Delta SA = 2\pi R\, \Delta \ell$$
$$= 2\pi x^3 \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

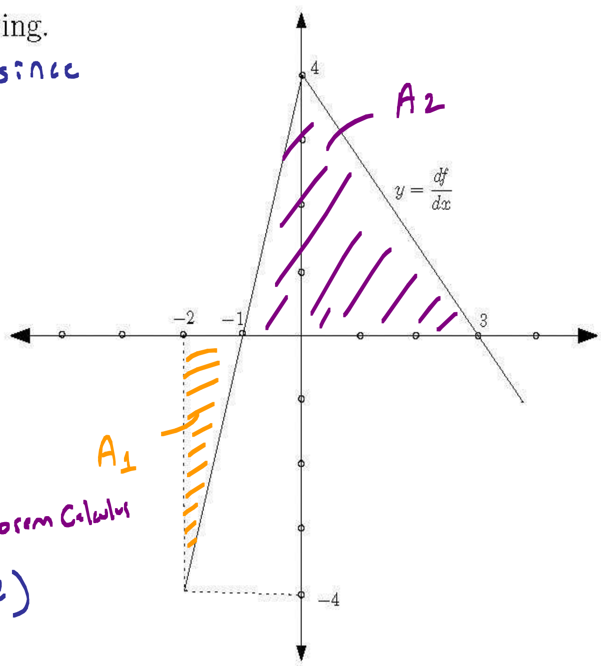
Hence
$f(x) = x^3$
$f'(x) = 3x^2$

$$SA = 2\pi \int_1^2 x^3 \sqrt{1 + (f'(x))^2}\, dx$$

$$= 2\pi \int_1^2 x^3 \sqrt{1 + 9x^4}\, dx = \frac{2\pi}{36} \int_{10}^{145} u^{1/2}\, du = \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_{10}^{145} = \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$$

$u = 1 + 9x^4$
$du = 36x^3\, dx$

Question 8 (5+5+5 pts) The graph of $y = f'(x)$ is given below. Answer the following questions using the given graph.



(a) Find the intervals where $f(x)$ is increasing.

$f(x)$ is increasing on $[-1, 3]$ since
$f'(x) > 0$ on $(-1, 3)$

(b) Evaluate $\int_{-2}^{3} f'(x)\, dx$. $= A_2 - A_1$

$$= \frac{1}{2} 4 \cdot 4 - \frac{1}{2} 1 \cdot 4$$

$$= 8 - 2 = 6$$

(c) If $f(-2) = 113$ find $f(3)$.

by Fundemental Theorem Calculus

$$6 = \int_{-2}^{3} f'(x)\, dx = f(3) - f(-2)$$
$$= f(3) - 113$$
$$\Rightarrow \quad f(3) = 6 + 113 = 119$$

M E T U Department of Mathematics

Math 119 Calculus Exam 1 07.04.2012		
Last Name : Name : Student No:	Signature : Duration : 90 *minutes*	Section
5 QUESTIONS ON 4 PAGES		TOTAL 90 POINTS

1	2	3	4	5			
					KEY		

YOU ARE NOT ALLOWED TO USE L'HOPITAL'S RULES IN THIS EXAM!
SHOW YOUR WORK!

(8+8+8 pts) **1.** Evaluate the limit or explain why it does not exist:

(a) $\lim_{x\to 1}\frac{|x^2-5|-|x+3|}{x-1} = \lim_{x\to 1}\frac{-(x^2-5)-(x+3)}{x-1} = \lim_{x\to 1}\frac{-(x^2+x-2)}{x-1}$

$= \lim_{x\to 1}\frac{-(x+2)(x-1)}{x-1} = \lim_{x\to 1} -(x+2) = -3$

(b) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

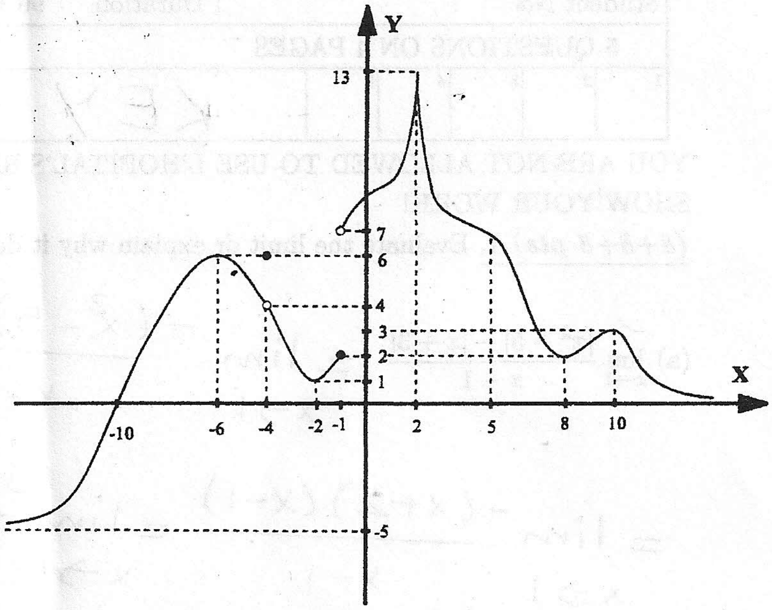
$= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\frac{1}{1+\cos x} = 1^2\cdot\frac{1}{1+1} = \frac{1}{2}$

(c) $\lim_{x\to\infty}(\sqrt{x^2+4x}-\sqrt{x^2-4x}) = \lim_{x\to\infty}\frac{(\sqrt{x^2+4x}-\sqrt{x^2-4x})(\sqrt{x^2+4x}+\sqrt{x^2-4x})}{\sqrt{x^2+4x}+\sqrt{x^2-4x}}$

$= \lim_{x\to\infty}\frac{x^2+4x-(x^2-4x)}{\sqrt{x^2(1+\frac{4}{x})}+\sqrt{x^2(1-\frac{4}{x})}} = \lim_{x\to\infty}\frac{8x}{x(\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{4}{x}})}$

$= \lim_{x\to\infty}\frac{8}{\sqrt{1+\frac{4}{x}}+\sqrt{1-\frac{4}{x}}} = \frac{8}{\sqrt{1+0}+\sqrt{1-0}} = 4$

(2 pts each=18 pts) 2. For the function $f(x)$ whose graph is given, find the followings:



(a) $\lim_{x \to 2} f(x) =$ 13

(b) $\lim_{x \to -4} f(x) =$ 4

(c) $\lim_{x \to -\infty} f(x) =$ -5

(d) $\lim_{x \to -1} f(x) =$ does not exist

(e) Points where $f'(x) = 0$. $x = -6, -2, 8, 10$

(f) Intervals where $f'(x) > 0$. $(-\infty, -6), (-2, -1), (-1, 2), (8, 10)$

(g) Intervals where f is continuous. $(-\infty, -4), (-4, -1), (-1, \infty)$

(g) Intervals where f is differentiable. $(-\infty, -4), (-4, -1), (-1, 2), (2, \infty)$

(h) $\lim_{x \to 2^-} f'(x) =$ ∞

(10+10 pts) 3. Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$

(a) Show that the function $f(x)$ is continuous at $x = 0$.

$f(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

So $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous at $x=0$.

(b) Use the definition of derivative to determine whether $f'(0)$ exists.

$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$ $\quad (f(0)=0)$

$\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0$

$\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0^-} \sin\left(\frac{1}{h}\right)$ does not exist.

So $f'(0)$ does not exist.

(9+7 pts) **4.** Let f and g be differentiable functions such that $f(9) = -2$, $g(3) = 8$, $f'(9) = 7$, $g'(3) = 2$.

(a) Show that $y' = -11$ at the point $(3, 1)$ if $yg(x) + f(x^2) = y^4 + 5$.

$$\frac{d}{dx}\left(yg(x) + f(x^2)\right) = \frac{d}{dx}\left(y^4 + 5\right)$$

$$y'g(x) + yg'(x) + 2xf'(x^2) = 4y^3 y'$$

Substitute $x = 3$, $y = 1$:

$$y'g(3) + g'(3) + 6f'(9) = 4y'$$

$$8y' + 2 + 42 = 4y', \quad 4y' = -44, \quad y' = -11.$$

(b) Find an equation of the line tangent to the curve $yg(x) + f(x^2) = y^4 + 5$ at the point $(3, 1)$.

$m = -11$, point $= (3, 1)$

$$y = -11(x-3) + 1$$

(12 pts) **5.** If $f(1) = -2$ and $3 \le f'(x) \le 5$ for all x, show that $1 \le f(2) \le 3$.

Since $f'(x)$ exists for all x, f is differentiable and hence continuous for all x. So we can apply MVT to $f(x)$ on $[1, 2]$:

$$\frac{f(2) - f(1)}{2 - 1} = f'(c) \quad \text{for some } c,\ 1 < c < 2.$$

$$3 \le \frac{f(2) - (-2)}{1} \le 5, \quad 1 \le f(2) \le 3.$$

M E T U Department of Mathematics

Math 119 Calculus Exam 2 05.05.2012		
Last Name : Name : Student No:	Signature : Duration : 100 *minutes*	Section
5 QUESTIONS ON 4 PAGES		TOTAL 90 POINTS

1	2	3	4	5	KEY		

SHOW YOUR WORK!

(6+6+6+6 pts) **1.** Evaluate the followings:

(a) $\frac{d}{dx}\left((\tan x)^x\right)$ (DO NOT SIMPLIFY)

$$= \frac{d}{dX}\left(e^{X\ln(\tan X)}\right) = e^{X\ln(\tan X)} \cdot \left(\ln(\tan X) + X\,\frac{\sec^2 X}{\tan X}\right)$$

(b) $\lim\limits_{x\to\infty} \frac{5^x - 2^x}{5^{x+1} + 2^x}$

$$= \lim_{X\to\infty} \frac{1-\left(\frac{2}{5}\right)^X}{5+\left(\frac{2}{5}\right)^X} = \frac{1}{5} \qquad \left(\lim_{X\to\infty}\left(\frac{2}{5}\right)^X = 0\right)$$

(c) $\lim\limits_{x\to\infty} \left(\ln(1+x^2) - \ln(1+x)\right)$

$$= \lim_{X\to\infty} \ln\left(\frac{1+X^2}{1+X}\right) = \infty \qquad \left(\lim_{X\to\infty}\frac{1+X^2}{1+X} = \infty \text{ and } \lim_{X\to\infty} \ln X = \infty\right)$$

(d) $\lim\limits_{x\to 0^+} (\tan x)^{\frac{1}{\ln x}}$

$$= \lim_{X\to 0^+} e^{\frac{\ln(\tan X)}{\ln X}} = e^{\lim\limits_{X\to 0^+}\frac{\ln(\tan X)}{\ln X}} = e^1 = e$$

$$\left[\lim_{X\to 0^+} \frac{\ln(\tan X)}{\ln X} \underset{L'H}{\overset{\left[\frac{\infty}{\infty}\right]}{=}} \lim_{X\to 0^+} \frac{\frac{\sec^2 X}{\tan X}}{\frac{1}{X}} = \lim_{X\to 0^+} \underbrace{\frac{X}{\sin X}}_{\text{limit exists}} \cdot \underbrace{\frac{1}{\cos X}}_{\text{limit exists}} = 1.1 = 1\right]$$

(4+2+4+2+4+4+8 pts) **2.** Let $f(x)$ be function defined for all real numbers. Suppose that $f(x)$ satisfies the following conditions:

$f(x)$ is not continuous at $x = \pm 1$,
$f'(x)$ is not defined at $x = -2$,
$f'(2) = 0$, $f''(0) = f''(3) = 0$,
$\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 3$,
$\lim_{x \to -1^-} f(x) = -2$, $\lim_{x \to -1^+} f(x) = \infty$,
$\lim_{x \to 1^-} f(x) = -\infty$, $\lim_{x \to 1^+} f(x) = 4$,

x		-2		-1		0		1		2		3	
$f(x)$		-1		-2		0		4		1		2	
$f'(x)$	+	‖	−	‖	−		−	‖	−	0	+		+
$f''(x)$	+	‖	+	‖	+	0	−	‖	+		+	0	−

Answer the following questions using the given information:

(a) Find all vertical and horizontal asymptotes of $f(x)$.

$\lim_{x \to \infty} f(x) = 3$, So, $y = 3$ is horizontal asymptote.

$\lim_{x \to -1^+} f(x) = \infty$, $\lim_{x \to 1^-} f(x) = -\infty$, So, $x = \pm 1$ are vertical asymptotes.

(b) Give all the intervals where $f(x)$ is increasing.

From the table : $(-\infty, -2)$ and $(2, \infty)$.

(c) Is $f(x)$ decreasing on $(-2, 2)$? Explain.

No! 0 and 1 are in $(-2, 2)$, $0 < 1$ but
$f(0) = 0 < 4 = f(1)$
($f(x)$ is not continuous at $x = \pm 1$.)

(d) Give all the intervals where $f(x)$ is concave down.

From the table: $(0, 1)$ and $(3, \infty)$.

(e) Find all local maximum and local minimum points of $f(x)$.

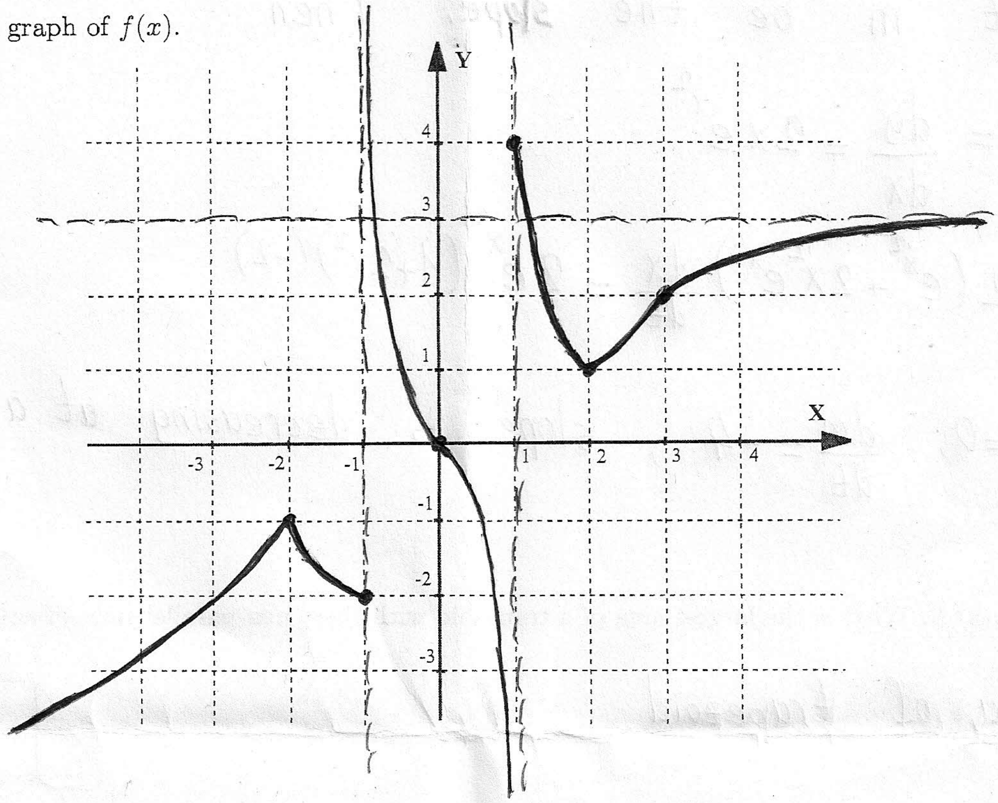
From the table; $f(-2) = -1$ is local maximum.
$f(2) = 1$ is local minimum.

Also, since $f(x)$ is decreasing on $(-2, -1)$ and $\lim_{x \to -1^+} f(x) = \infty$,
$f(-1) = -2$ is local minimum.
Similarly, since $f(x)$ is decreasing on $(1, 2)$ and
$\lim_{x \to 1^-} f(x) = -\infty$, $f(1) = 4$ is local maximum.

(f) Find all inflection points of $f(x)$.

$f(0)=0$, $f(3)=2$ are inflection points (from the table).
at $x=1$ $f(x)$ is not continuous, so $x=1$ is not an inflection point.

(g) Sketch a graph of $f(x)$.



(6+6 pts) **3.** Let $f(x) = 3 + x + e^x$.
(a) Show that $f(x)$ has an inverse function $f^{-1}(x)$.

$f'(x) = 1 + e^x > 1 > 0$. So, $f(x)$ is increasing and hence 1-1 for all x. So, it has an inverse function $f^{-1}(x)$.

(b) Find $(f^{-1})'(4)$.

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}$$

$f(0)=4$. So, $f^{-1}(4)=0$ and $(f^{-1})'(4) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

(13 pts) **4.** A point (x, y) is moving on the curve $y = e^{x^2}$ so that its x-coordinate is decreasing at a rate of 2 units/sec. Find the rate of change of the slope of the tangent line to the curve $y = e^{x^2}$ at the moment when $x = 0$.

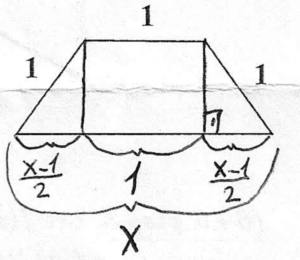
Let m be the slope. Then,

$$m = \frac{dy}{dx} = 2xe^{x^2}$$

$$\frac{dm}{dt} = 2\left(e^{x^2} + 2x^2 e^{x^2}\right)\frac{dx}{dt} = 2e^{x^2}(1+2x^2)(-2)$$

when $x=0$, $\frac{dm}{dt} = -4$, slope is decreasing at a rate of 4 units/sec²

(13 pts) **5.** What is the largest area of a trapezoid with three non-parallel sides of equal length 1?



A = area of trapezoid

$$A = \frac{(x+1)\,h}{2}$$

$$h = \sqrt{1 - \left(\frac{x-1}{2}\right)^2} = \frac{\sqrt{3-x^2+2x}}{2}$$

$$A(x) = \frac{(x+1)\sqrt{3-x^2+2x}}{4}, \quad 0 \leq x \leq 3$$

$$A(0) = \frac{\sqrt{3}}{4}, \quad A(3) = 0.$$

$$\frac{dA}{dx} = \frac{1}{4}\left(\sqrt{3-x^2+2x} + (x+1)\frac{-2x+2}{2\sqrt{3-x^2+2x}}\right) = \frac{1}{4\sqrt{3-x^2+2x}}\left(3-x^2+2x-x^2-x+x+1\right)$$

$$\frac{dA}{dx} = \frac{-(x-2)(x+1)}{2\sqrt{3-x^2+2x}} = 0 \iff x=2 \rightsquigarrow \text{critical point.}$$

$3-x^2+2x = -(x-3)(x+1) = 0 \iff x=3 \rightsquigarrow$ singular point. $(A(3)=0)$

$A(2) = \frac{3\sqrt{3}}{4} > \frac{\sqrt{3}}{4}$. So, $\frac{3\sqrt{3}}{4}$ is maximum area.

M E T U Department of Mathematics

Math 119 Calculus I Final Exam 31.05.2012		
Last Name: Name : Student No:	Signature : Duration : 120 *minutes*	Section
6 QUESTIONS ON 6 PAGES		TOTAL 120 POINTS

1	2	3	4	5	6	KEY		

SHOW YOUR WORK!

1.

(5+5 pts) a) Evaluate the limit or explain why it does not exist:

(i) $\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x}$

$$= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{\sin^2 x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin^2 x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1}{1+\cos x} = \frac{1}{2}$$

(i) $\lim_{x\to 3} \frac{(x^2+x-12)\sqrt{x+3}}{\sqrt{5}(x^2-9)}$

$$= \lim_{x\to 3} \frac{(x+4)(x-3)\sqrt{x+3}}{\sqrt{5}(x-3)(x+3)}$$

$$= \lim_{x\to 3} \frac{(x+4)\sqrt{x+3}}{\sqrt{5}(x+3)}$$

$$= \frac{7\sqrt{6}}{6\sqrt{5}} = \frac{7}{\sqrt{30}}$$

(5 pts) b) Use the definition of the derivative to find the derivative function of $f(x) = \sqrt{2x}$. (DO NOT USE L'HOPITAL'S RULE!)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}$$

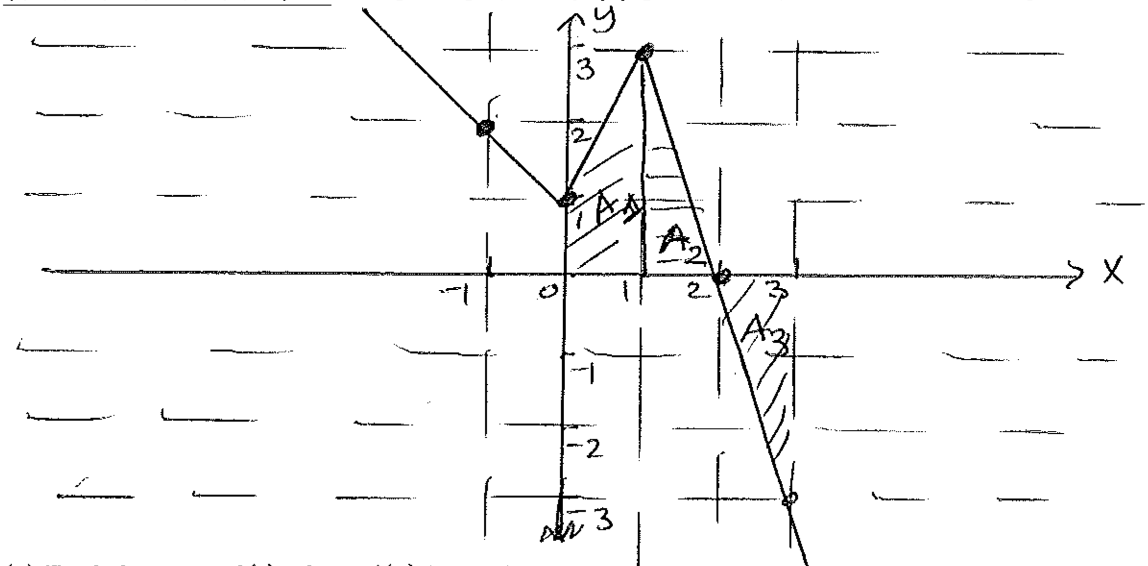
$$= \lim_{h\to 0} \frac{(\sqrt{2x+2h}-\sqrt{2x})(\sqrt{2x+2h}+\sqrt{2x})}{h(\sqrt{2x+2h}+\sqrt{2x})}$$

$$= \lim_{h\to 0} \frac{2x+2h-2x}{h(\sqrt{2x+2h}+\sqrt{2x})} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h}+\sqrt{2x}} = \frac{2}{2\sqrt{2x}} = \frac{1}{\sqrt{2x}}$$

(5 pts) c). Use differentiation formulas to find the derivative function of $f(x) = \ln(x^2 + e^\pi)$.

$$f'(x) = \frac{2x}{x^2+e^\pi}$$

(4+4+4+4+4+4 pts) **2.** Using the graph of $f'(x)$ given below, answer the following questions.



(a) Find the interval(s) where $f(x)$ is continuous.

Since $f'(x)$ exists for all x in $(-\infty, \infty)$, $f(x)$ is continuous on $(-\infty, \infty) = \mathbb{R}$.

(b) Find the interval(s) where $f(x)$ is decreasing.

$f'(x) < 0 \Rightarrow f(x)$ is decreasing.
So $f(x)$ is decreasing on $[2, \infty)$.

(c) Find the interval(s) where the graph of $f(x)$ is concave up.

$f(x)$ is concave up when $f'(x)$ is increasing.
So $f(x)$ is concave up on $(0, 1)$.

(d) Find the point(s) where $f''(x)$ does not exist.

$x = 0$ and $x = 1$. (At these points, graph of $f'(x)$ has corners.)

(e) Evaluate $\int_0^3 f'(x)\, dx$. $= A_1 + A_2 - A_3$

$= \frac{1}{2}(1+3)1 + \frac{1}{2}3 \cdot 1 - \frac{1}{2}3 \cdot 1 = 2$

(f) If $f(0) = -1$, find $f(3)$ using part (e).

$2 = \int_0^3 f'(x)\,dx \underset{(F.T.C.)}{=} f(3) - f(0) = f(3) - (-1)$

$f(3) = 2 - 1 = 1$

(8+8+8 pts) **3.** Find the following integrals:

(a) $\displaystyle\int_0^{\pi/4} \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = I.$

Let $u = \sin x + \cos x$.
Then $du = (\cos x - \sin x)\,dx$
$= -(\sin x - \cos x)\,dx$
$x = 0 \Rightarrow u = 1$, $x = \frac{\pi}{4} \Rightarrow u = \sqrt{2}$

so $I = \displaystyle\int_1^{\sqrt{2}} -\frac{du}{u}$

$= -\ln u \Big|_1^{\sqrt{2}} = -\ln\sqrt{2} = -\dfrac{\ln 2}{2}$.

(b) $\displaystyle\int (x-1)^2 \ln(x-1)\,dx = I.$ Let $u = \ln(x-1)$, $dv = (x-1)^2\,dx$.
Then $du = \dfrac{dx}{x-1}$, $v = \dfrac{(x-1)^3}{3}$.

So $I = \dfrac{(x-1)^3}{3}\ln(x-1) - \dfrac{1}{3}\displaystyle\int (x-1)^2\,dx$

$= \dfrac{(x-1)^3}{3}\ln(x-1) - \dfrac{(x-1)^3}{9} + C$

(c) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx = I.$ $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

So $A(x^2+9) + (Bx+C)(x-1) = 10$.

Hence $A + B = 0$, $-B + C = 0$, $9A - C = 10$.

$A = -B = -C$. So $-10C = 10$
$C = -1$, $A = 1$
$B = -1$.

so
$I = \displaystyle\int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9} \right) dx$

$= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{9}\displaystyle\int \frac{dx}{(\frac{x}{3})^2+1} = \ln\dfrac{|x-1|}{\sqrt{x^2+9}} - \dfrac{1}{3}\arctan\left(\dfrac{x}{3}\right) + C$

(10 pts) **4.** Let $f(x) = \int_1^x \frac{dt}{1+t^6}$. Use the linearization of $f(x)$ at $a = 1$ to approximate $f\left(\frac{3}{2}\right)$.

$f(1) = 0$. $f'(x) = \frac{1}{1+x^6}$, $f'(1) = \frac{1}{2}$

$L(x) = \frac{1}{2}(x-1)$.

$f\left(\frac{3}{2}\right) \approx L\left(\frac{3}{2}\right) = \frac{1}{2}\left(\frac{3}{2} - 1\right) = \frac{1}{4}$

(8+8 pts) **5.** Determine whether the given improper integral is convergent or divergent. Explain your reason(s). State the hypothesis and conclusion of any theorems you use.

(a) $\int_0^\infty xe^{-x^2}\,dx = \lim_{R\to\infty} \int_0^R xe^{-x^2}\,dx = \lim_{R\to\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_0^R\right)$

$= \lim_{R\to\infty}\left(-\frac{1}{2}e^{-R^2} + \frac{1}{2}\right) = \frac{1}{2}$ convergent.

(b) $\int_0^{\pi/2} \frac{dx}{x\sin x}$

For $0 < x \le \pi/2$, $f(x) = \frac{1}{x\sin x} > 0$

and $0 < \sin x \le 1$. So $0 < x\sin x \le x$

and $\frac{1}{x\sin x} \geq \frac{1}{x} > 0$. $\int_0^{\pi/2} \frac{dx}{x}$ is divergent with p-test $(p=1)$. Comparison Theorem says if $f(x) \geq g(x) > 0$ for x in $[a,b]$ and $\int_a^b g(x)dx$ is divergent, then $\int_a^b f(x)dx$ is divergent. So, by comparison $\int_0^{\pi/2} \frac{dx}{x\sin x}$ is divergent.

NAME: ______________________ SIGNATURE:

6. Let R be the region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$.

(4 pts) **a.** Sketch and shade the region R.

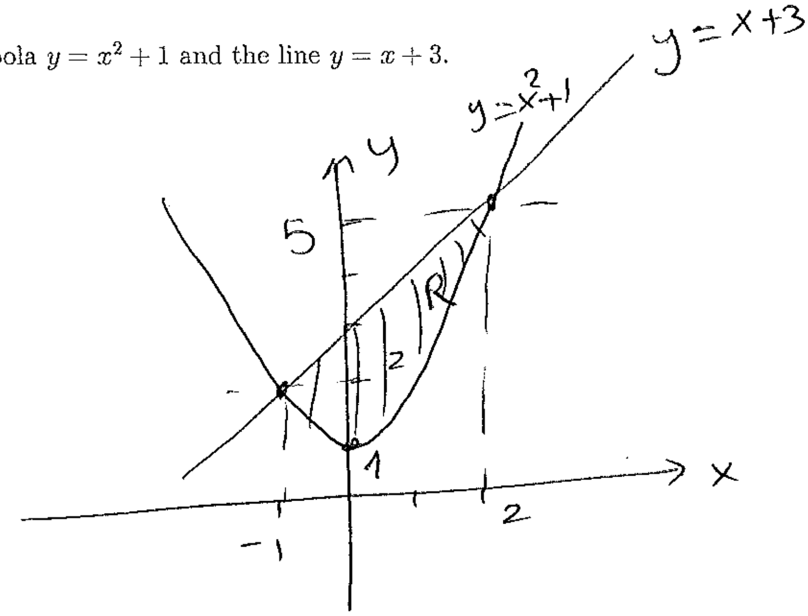
$x^2+1 = x+3$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

$y = 5, 2$



(8 pts) **b.** Find the area of the region R.

$$\text{area of } R = A = \int_{-1}^{2} \left(x+3-(x^2+1)\right)dx$$

$$= \int_{-1}^{2} (x+3-x^2-1)\,dx$$

$$= \int_{-1}^{2} (-x^2+x+2)\,dx = -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

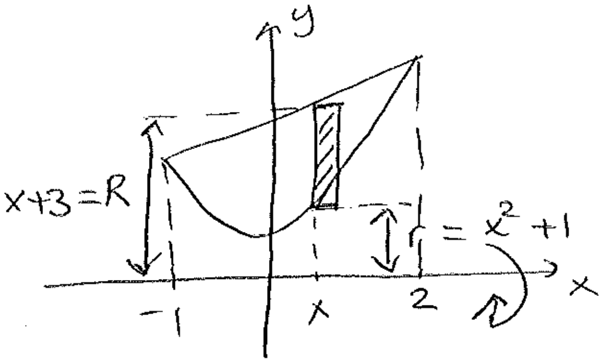
$$= -\frac{8}{3} + 2 + 4 - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

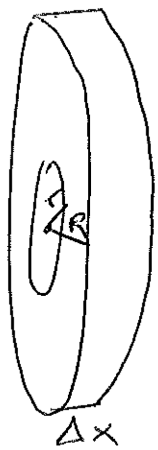
$$= -\frac{8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2 = -3 + 8 - \frac{1}{2} = 5 - \frac{1}{2}$$

$$= \frac{9}{2}$$

(7+7 pts) **c.** Let S be the solid obtained by rotating R around the $x-$axis. In each part sketch the region R, draw a typical rectangle (area element dA) and a typical volume (volume element dV) and use them to write a definite integral (or a sum of definite integrals, DO NOT EVALUATE THE INTEGRALS) which gives the volume of S using

(i) slicing (disc) method,





volume

$= \pi (R^2 - r^2)\, \Delta x$

$= \pi \left((x+3)^2 - (x^2+1)^2 \right) \Delta x$

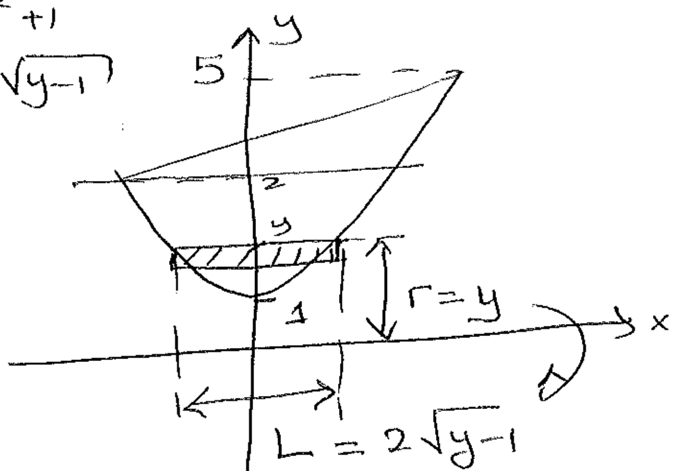
$$V = \int_{-1}^{2} \pi \left((x+3)^2 - (x^2+1)^2 \right) dx$$

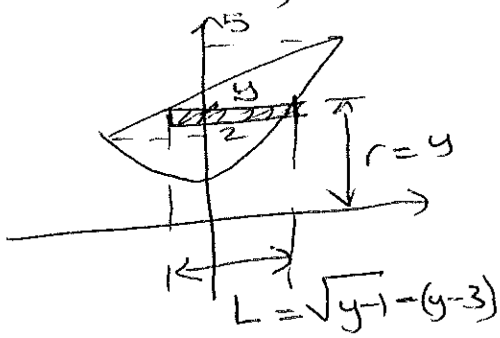
$$= \int_{-1}^{2} \pi \left(-x^2 + 6x + 8 - x^4 \right) dx$$

(ii) cylindrical shell method.

$y = x^2 + 1$

$x = \mp \sqrt{y-1}$





For $2 \le y \le 5$

volume

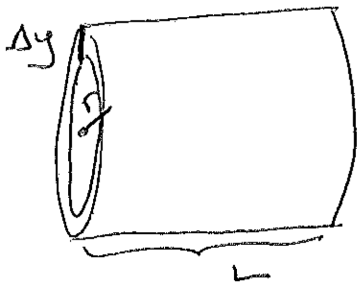
$= 2\pi r L\, \Delta y$

$= 2\pi y \left(\sqrt{y-1} - y + 3 \right) \Delta y$

For $1 \le y \le 2$

volume $= 2\pi r L\, \Delta y$

$= 2\pi y\, 2\sqrt{y-1}\, \Delta y$



$$V = \int_{1}^{2} 4\pi y \sqrt{y-1}\, dy + \int_{2}^{5} 2\pi y \left(\sqrt{y-1} - y + 3 \right) dy$$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANAYTIC GEOMETRY MidTerm 2	List No.
Code : *Math 119* Acad. Year : *2011-2012* Semester : *Fall* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :	
Date : *December.17.2011* Time : *9:30* Duration : *119 minutes*	5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	SHOW YOUR WORK	

Question 1 (20 pts)

Let $y(x)$ be a function defined implicitly by the equation $x^2y^3 - x^3y^2 = 4$.

a) Find $y'(x)$ at $(1,2)$.

$x^2 y^3(x) - x^3 y^2(x) = 4$; taking derivative of both sides with respect to x, we obtain;

$2xy^3(x) + 3x^2y^2(x)\,y'(x) - 3x^2y^2(x) - 2x^3 y(x)\,y'(x) = 0$

At the point $(x, y(x)) = (1,2)$ we have

$2\cdot 8 + 3\cdot 4\, y'(1) - 3\cdot 4 - 2\cdot 2\, y'(1) = 0$

$\Rightarrow \quad y'(1) = \dfrac{-4}{8} = \dfrac{-1}{2}$

b) Using the linearization (tangent line approximation) of $y(x)$ at the point $(1,2)$ approximate the value of $y(1.01)$.

$y(x) \approx L_y(x) = y'(a)(x-a) + y(a)$ for x close to a. Hence

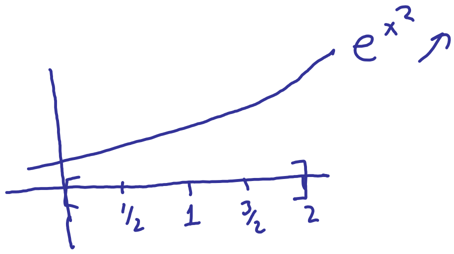
$y(1{,}01) \approx y(1) + y'(1)(1.01 - 1)$

$= 2 - \frac{1}{2}\,\frac{1}{100} = 2 - \frac{1}{200} = \frac{399}{200}$

Question 2 (16 pts)

a) Write upper and lower Riemann sums for $y = e^{x^2}$ on $[0, 2]$ corresponding to 4 equal length subintervals of $[0, 2]$.

Since $f(x)$ is an increasing function, lower sum of f with respect to partition $\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ is the same as left sum. Similarly, upper sum is the same as right sum. Hence; using $\Delta x_i = \frac{2-0}{4} = \frac{1}{2}$



$\mathcal{P} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

$$L_f(\mathcal{P}) = \left[f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\right]\tfrac{1}{2}$$
$$= \left[e^0 + e^{1/4} + e^1 + e^{9/4}\right]\tfrac{1}{2}$$

$$U_f(\mathcal{P}) = \left[f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right]\tfrac{1}{2}$$
$$= \left[e^{1/4} + e^1 + e^{9/4} + e^4\right]\tfrac{1}{2}$$

b) Using part (a) prove that $\int_0^2 e^{x^2}\,dx > 4$.

For any partition $\mathcal{P}$ of the interval $[0,2]$ we have $L_f(\mathcal{P}) \le \int_0^2 e^{x^2}dx \le U_f(\mathcal{P})$

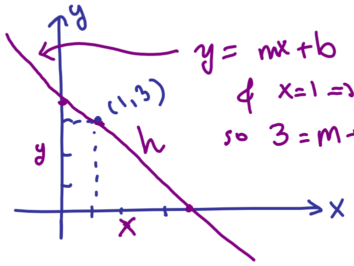
In particular for the partition $\mathcal{P}$ in part (a) we have

$$\int_0^2 e^{x^2}dx \ge L_f(\mathcal{P}) = \tfrac{1}{2}\left[e^0 + e^{1/4} + e^1 + e^{9/4}\right].$$
$$\overset{*}{>} \tfrac{1}{2}\left[1 + 1 + 2 + 4\right]$$
$$= 4$$

$*$ Using the fact that $e \approx 2.71 > 2$ we obtain
$e^0 = 1$, $e^{1/4} > 1$
$e^1 > 2$, $e^{9/4} > e^2 > 4$

Question 3 (16 pts) Among all triangles formed by the positive x-axis, the positive y-axis and a line passing through the point $(1,3)$, find the dimensions of the triangle with the shortest hypotenuse.

we are asked to find x, y & $h = \sqrt{x^2+y^2}$ with h-minimum (or, equivalently x^2+y^2 is minimum)



$y = mx + b$
& $x = 1 \Rightarrow y = 3$
so $3 = m + b$

$y = mx + (3-m)$

Notice that x & y are intercepts of the line $y = mx + (3-m)$.
hence $x = 0 \Rightarrow y = 3 - m$
$y = 0 \Rightarrow x = \frac{m-3}{m} = 1 - \frac{3}{m}$

Thus: we want to find $m \in (-\infty, 0)$ which minimizes the continuous function
$f(m) = (3-m)^2 + (1 - \frac{3}{m})^2$

Since $\lim_{m\to-\infty} f(m) = +\infty$
and $\lim_{m\to 0^-} f(m) = +\infty$
and f is diff'ble $(-\infty, 0)$

min of f on $(-\infty, 0)$ exists and it is at one of the critical point.

$$f'(m) = \underbrace{-2(3-m)}_{2(m-3)} + 2\underbrace{(1-\tfrac{3}{m})}_{(m-3)}\left(\tfrac{3}{m^2}\right) = (m-3)\left(2 + \tfrac{6}{m^3}\right) = 0 \Rightarrow m_1 = 3 \notin (-\infty, 0)$$
$$\Rightarrow \tfrac{3}{m^3} = -1 \Rightarrow m = -3^{1/3}.$$

Hence $m = -3^{1/3}$ is the only critical point in $(-\infty, 0)$ hence it gives the minimum. For this m, dimensions of the triangle are: $x = 1 - \frac{3}{-3^{1/3}} = 1 + 3^{2/3}$ and $h = \sqrt{x^2+y^2} = (1 + 3^{2/3})^{3/2} = h$

$y = 3 + 3^{1/3}$

Question 4 (28 pts) Evaluate the following

a) $\displaystyle\int_0^2 x^2 e^{x^3}\,dx$

$$= \int_0^8 \frac{1}{3}e^u\,du = \frac{1}{3}e^u\Big|_0^8 = \frac{1}{3}(e^8-1)$$

$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$

$x=0 \Rightarrow u=0$

$x=2 \Rightarrow u=8$

b) $\displaystyle\lim_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^x$

$$= e^{\lim\limits_{x\to\infty} \overset{\infty}{x}\, \overset{0}{\ln\cos\frac{2}{x}}} = e^{\lim\limits_{x\to\infty} \frac{\ln\cos\frac{2}{x}\to 0}{\frac{1}{x}\to 0}} \quad \left(\frac{0}{0}\text{ type}\right)$$

L'Hop's Rule

$$= e^{\lim\limits_{x\to\infty} \frac{(\sin\frac{2}{x})\left(\frac{2}{\cancel{x^2}}\right)}{(\cos\frac{2}{x})\left(\frac{-1}{\cancel{x^2}}\right)}} = e^{\lim\limits_{x\to\infty} \frac{-2\sin\frac{2}{x}\to 0}{\cos\frac{2}{x}\to 1}} = e^0 = 1$$

c) $\displaystyle\lim_{x\to 0^+}\frac{238x + x^{2011}}{\int_0^x(\cos(t^3)+\cos^3 t)dt}$

$\left(\frac{0}{0}\text{ type}\right) \Rightarrow$ we can use L'Hop's Rule;

$$= \lim_{x\to 0^+}\frac{238 + 2011x^{2010}}{\cos x^3 + \cos^3 x} = \frac{238}{\cos 0 + \cos^3 0} = \frac{238}{2} = 119$$

d) $\displaystyle\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n}\sqrt{1-\frac{i^2}{n^2}}$

since each limit exists

$$\overset{=}{=} \underbrace{\lim_{n\to\infty}\frac{1}{n}\sqrt{1-\frac{0^2}{n^2}}}_{=0} + \underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\frac{i^2}{n^2}}}_{=\pi/4} = \pi/4$$

Let P be the partition of the interval $[0,1]$ into n equal length subintervals

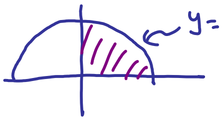
$\Delta x_i = 1/n$

$0 \quad 1/n \quad 2/n \quad \cdots \quad \frac{n-1}{n} \quad 1$

and $f(x) = \sqrt{1-x^2}$

Then $$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\frac{1}{n}}_{=\Delta x_i}\underbrace{\sqrt{1-\left(\frac{i}{n}\right)^2}}_{=f(x_i^*)} = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i^*)\,\Delta x_i = \int_0^1\sqrt{1-x^2} = \text{Area} = \frac{1^2\cdot\pi}{4} = \frac{\pi}{4}$$

where x_i^* is choosen as right end point of i^{th} subinterval



Question 5 (20 pts) Let $f(x) = \dfrac{x^3 - 3x^2 + 1}{x^3}$.

a) Find the horizontal and vertical asymptotes of f.

H.A.: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\cancel{x^3}\left(1 - \frac{3}{x} + \frac{1}{x^3}\right)}{\cancel{x^3}} = 1 \Rightarrow \boxed{y=1}$ is HA.

V.A: candidate is $x=0$: since $\lim_{x\to 0^{\pm}} f(x) = \pm\infty$, $\boxed{x=0}$ is V.A.

b) Find the intervals on which f is increasing, and the intervals on which f is decreasing.

$f(x) = 1 - \frac{3}{x} + \frac{1}{x^3} \Rightarrow f'(x) = \frac{3}{x^2} - \frac{3}{x^4} = 0 \Rightarrow \frac{1}{x^2} = \frac{1}{x^4} \Rightarrow x^2 = x^4 \quad (x \neq 0)$

$\Rightarrow x = \mp 1$ are the only critical points

x		-1		0		$+1$	
f'	$+$	0	$-$	$\parallel$	$-$	0	$+$
f	↗		↘	$\parallel$	↘		↗

f is increasing on $(-\infty, -1)$ and on $(1, +\infty)$

f is decreasing on $(-1, 0)$ and on $(0, +1)$.

c) Find the points where f has a local maximum and the points where f has a local minimum.

since the sign of f' changes from $+$ to $-$ through $x=-1$, we have $x=-1$ is a local MAXIMUM point

similarly $x=+1$ is a local minimum point

d) Find the intervals of concavity and inflection point(s) of f.

$f''(x) = 3\left(-2x^{-3} + 4x^{-5}\right) = 3\left(\frac{-2}{x^3} + \frac{4}{x^5}\right)$

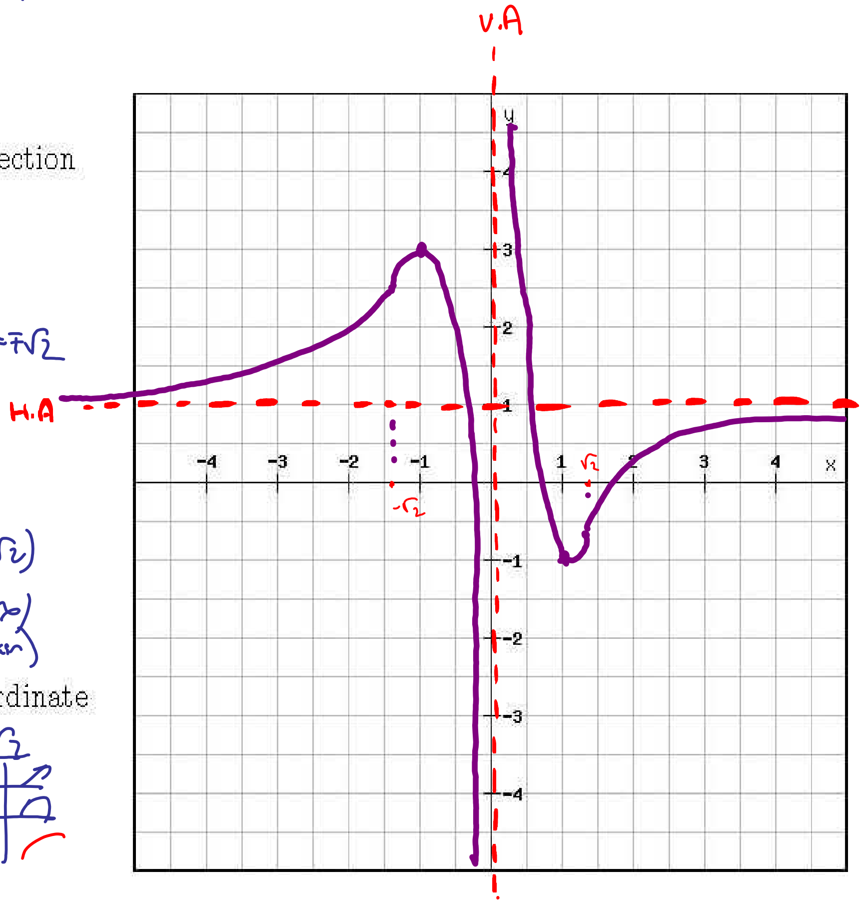
$= 0 \Rightarrow \frac{1}{x^3} = \frac{2}{x^5} \overset{x\neq 0}{\Rightarrow} 1 = \frac{2}{x^2} \Rightarrow x = \mp\sqrt{2}$

x		$-\sqrt{2}$		0		$+\sqrt{2}$	
f''	$+$	0	$-$	$\parallel$	$+$	0	$-$
f	$\cup$		$\cap$	$\parallel$	$\cup$		$\cap$

f is concave up on $(-\infty, -\sqrt{2})$ and on $(0, +\sqrt{2})$

" " " down on $(-\sqrt{2}, 0)$ and on $(+\sqrt{2}, +\infty)$

$\mp\sqrt{2}$ are inflection points (0 is not in the domain)



e) Sketch the graph of f on the given coordinate plane.

x		$-\sqrt{2}$		-1		0		1		$\sqrt{2}$	
f	↗		↗		↘		↘		↗		↗
f	$\cup$		$\cap$		$\cap$		$\cup$		$\cup$		$\cap$
f											

M E T U Department of Mathematics

	Math 119 Calculus With Analytic Geometry Exam 1	

Acad. Year : *2015-2016* Semester : *Spring* Coordinator: *Emre Coşkun* Date : *April/9/2016* Time : *13:30* Duration : *90 minutes*	Last Name : Name : Student No. : Department : Section : Signature :
	6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

(8+8+8 pts) **1.** Evaluate each of the following limits if it exists.

(a) $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-x-4}\right) = \lim\limits_{x\to-\infty} \dfrac{(x+\sqrt{x^2-x-4})(x-\sqrt{x^2-x-4})}{(x-\sqrt{x^2-x-4})}$

$= \lim\limits_{x\to-\infty} \dfrac{x+4}{x-\underbrace{|x|}_{"-"}\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}} = \lim\limits_{x\to-\infty} \dfrac{x+4}{x+x\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}} = \lim\limits_{x\to-\infty} \dfrac{1+4/x}{1+\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}} = \dfrac{1}{2}$

(b) $\lim\limits_{x\to0} x\sin(x) = 0$

note that $\lim\limits_{x\to0} x = 0$ and $\lim\limits_{x\to0}\sin x = 0$, and hence

$$\lim_{x\to0}(x\sin x) = \underbrace{\lim_{x\to0} x}_{0} \cdot \underbrace{\lim_{x\to0}\sin x}_{0} = 0$$

both limit exist

(c) $\lim\limits_{x\to0} \dfrac{x^2\sin\left(\frac{1}{x}\right)}{\sin(x)}$

$= \lim\limits_{x\to0} \dfrac{x}{\sin x} \cdot \dfrac{\sin(1/x)}{1/x} = 1.0 = 0$

since $\lim\limits_{x\to0} \dfrac{x}{\sin x} = 1$ and $\lim\limits_{x\to0} \dfrac{\sin(1/x)}{1/x} = 0$

or, $-\dfrac{x^2}{|\sin(x)|} \le \dfrac{x^2\sin(1/x)}{\sin x} \le \dfrac{x^2}{|\sin x|} \Rightarrow \lim\limits_{x\to0} \dfrac{x^2\sin(1/x)}{\sin x} = 0$

$-\dfrac{x^2}{|\sin(x)|} \to 0$ as $x\to0$, $\dfrac{x^2}{|\sin x|} \to 0$ as $x\to0$

(12 pts) 2. Find an equation of the normal line to the curve $x^3 + y^3 - 9xy = 0$ at the point $(4, 2)$.

By implicit differentiation, we have

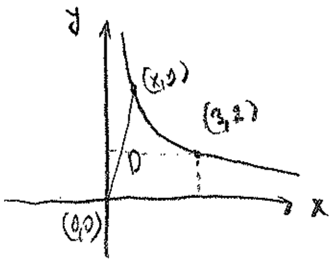
$$3x^2 + 3y^2y' - 9y - 9xy' = 0 \implies y' = \frac{3y - x^2}{y^2 - 3x}$$ whenever it exists.

Then, $$y'\Big|_{(4,2)} = \frac{3.2 - 16}{4-12} = \frac{-10}{-8} = \frac{5}{4}$$ a slope of tangent line through the point $(4,2)$.

So, the slope of the normal line is $-\frac{4}{5}$.

So, the eqn is $\boxed{y - 2 = -\frac{4}{5}(x-4)}$

(12 pts) 3. A point is moving to the right along the first-quadrant portion of the curve $x^2y^3 = 72$. When the point has coordinates $(3, 2)$, its horizontal velocity is 2 units/second. What is the rate of change in the distance of the particle to the origin?



Let $x = x(t)$ and $y = y(t)$ be coordinate functions parametrized by time t. so that $x^2y^3 = 72$. and $\frac{dx}{dt}\Big|_{(3,2)} = 2$ units/sec.

Let $D(t) = \sqrt{x^2+y^2}$ be the distance of the particle to the origin. $(D \geq 0)$.

We need to find $\frac{dD}{dt}\Big|_{(x,y)=(3,2)}$

Differentiate $x^2y^3 = 72$ wrt t, then we have

$$2xy^3\frac{dx}{dt} + 3xy^2\frac{dy}{dt} = 0 \xrightarrow[\text{evaluate at }(3,2)]{} \frac{dy}{dt}\Big|_{(3,2)} = -\frac{8}{9}.$$

Differentiate $D^2 = x^2+y^2$ wrt t, then we have

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \xrightarrow[\text{evaluate at }(3,2)]{} \sqrt{13}\frac{dD}{dt}\Big|_{(3,2)} = 6 + \frac{-16}{9}$$

So, $$\boxed{\frac{dD}{dt}\Big|_{(3,2)} = \frac{38}{9\sqrt{13}}}$$

(15 pts) 4. Let $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ \cos(x) & \text{if } x > 0. \end{cases}$

Is f continuous at $x = 0$? Explain why or why not. If $f'(0)$ exists, determine its value; if not, explain why it does not exist.

① Need to check $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$.

$\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1 = \lim_{x\to 0^+} \cos(x) = f(0)$ $\Rightarrow$ f is cont. at $x=0$.

② Now, we need to check $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \doteq f'(0)$ exists or not.

- $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\frac{\sin(x)}{x} - 1}{x} = \lim_{x\to 0^-} \frac{\sin(x)-x}{x^2} \underset{[\frac{0}{0}]}{\overset{L'H}{=}} \lim_{x\to 0^-} \frac{\cos x - 1}{2x} \underset{[\frac{0}{0}]}{\overset{L'H}{=}} \lim_{x\to 0^-} \frac{-\sin x}{2} = 0$

- $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\cos x - 1}{x} \underset{L'H}{\overset{[\frac{0}{0}]}{=}} \lim_{x\to 0^+} \frac{-\sin x}{1} = 0$, so, $f'(0)$ exists and $f'(0) = 0$.

(10 pts) 5. Prove that the function $f(x) = x^3 + 5x + 2$ has a unique zero in $\mathbb{R}$.

Clearly f is cont everywhere since it is polynomial.

In particular, f is also cont. on any interval $I = [a,b] \subset \mathbb{R}$.

Take $I = [-1, 0]$, then we also have $f(0) = 2 > 0$ $f(-1) = -4 < 0$.

By IVT, there is some $c \in [-1, 0]$ st $f(c) = 0$.

so, f is at least one zero. ①

(take $c_1 < c_2$)

Now, assume that f has more than one root, say c_1 and c_2.

Notice that f is clearly cont. on $[c_1, c_2]$ and differentiable on (c_1, c_2) since it is polynomial. By Rolle's thm, there exists some $d \in (c_1, c_2)$ such that $f'(d) = 0$.

But, $f'(x) = 3x^2 + 5$ which has no real root. Therefore, assumption is false. i.e., f cannot have more than one root ②

By ① and ②, it means f has exactly one zero.

(9+9+9 pts) 6.

(a) Find $\frac{dy}{dx}$ if $x^y = y^x$

Take the logarithm of both sides: $\ln x^y = \ln y^x \Rightarrow y\ln x = x\ln y \quad (*)$

Take the derivative of both sides of $(*)$ wrt x, then

$$y'\ln x + y\cdot\frac{1}{x} = \ln y + x\cdot\frac{1}{y}\cdot y'$$

$$\Rightarrow \quad y' = \frac{\ln y - y/x}{\ln x - x/y}$$

(b) Find $\frac{dy}{dx}$ if $y = \sinh(x)^{\ln x}$

Take the log. of both sides: $\ln y = \ln x \,.\, \ln(\sinh(x))$

Take the derivative of both sides wrt x, then we have

$$\frac{y'}{y} = \frac{1}{x}\ln(\sinh(x)) + \ln x\,.\,\frac{\cosh(x)}{\sinh(x)}$$

$$\Rightarrow \quad y' = [\sinh(x)]^{\ln x}\left[\frac{\ln(\sinh(x))}{x} + \ln x\,.\,\frac{\cosh(x)}{\sinh(x)}\right]$$

(c) Find $\frac{d}{dx}f^{-1}(1)$ where $f(x) = x^5 + 6x^3 + x + 1$.

Let $f^{-1}(1) = a$, then we have $f(a) = 1$

i.e $\quad a^5 + 6a^3 + a + 1 = 1 \quad \Rightarrow \quad a = 0$

<u>Recall</u> $\quad (f^{-1})'(1) = \dfrac{1}{f'(\underbrace{f^{-1}(1)}_{=0})}$, so $(f^{-1})'(1) = \dfrac{1}{f'(0)}$

Since $f'(x) = 5x^4 + 18x^2 + 1$ and $f'(0) = 1$, we have

$$(f^{-1})'(1) = 1 \; /\!/$$

M E T U Department of Mathematics

Math 119 Calculus With Analytic Geometry Exam 2		
Acad. Year : 2015-2016 Semester : Spring Coordinator: Emre Coşkun Date : May 14, 2016 Time : 13:30 Duration : 90 minutes	Last Name : Name : Department : Signature :	Student No. : Section :
	5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	SHOW YOUR WORK	

(8+8+8 pts) 1. Let $f(x) = \frac{2x^2-3x}{x-2}$.

(a) Find the asymptotes and the intercepts of the graph of $f(x)$.

x-intercepts: $y=0 \Rightarrow 2x^2 = 3x \Rightarrow x=0$ or $x=\frac{3}{2}$ $\Big\}$ $\boxed{(0,0)\ \&\ (\frac{3}{2},0)}$

y-intercept: $x=0 \Rightarrow y=0$

$$\left.\begin{aligned}\lim_{x\to\infty}\frac{2x^2-3x}{x-2}&=\infty\\ \lim_{x\to-\infty}\frac{2x^2-3x}{x-2}&=-\infty\end{aligned}\right\}\Rightarrow\boxed{\text{No H.A}}$$

$$\left.\begin{aligned}\lim_{x\to2^-}\frac{2x^2-3x}{x-2}&=-\infty\\ \lim_{x\to2^+}\frac{2x^2-3x}{x-2}&=\infty\end{aligned}\right\}\Rightarrow\boxed{\begin{matrix}x=2\\ \text{V.A}\end{matrix}}$$

$$\begin{array}{r|l} 2x^2-3x & x-2 \\ \cline{2-2} -(2x^2-4x) & 2x+1 \\ \hline x & \\ -(x-2) & \\ \hline 2 & \end{array}$$

$$\frac{2x^2-3x}{x-2} = 2x+1+\frac{2}{x-2}$$

$$\lim_{x\to\pm\infty}\left(2x+1+\frac{2}{x-2}\right)-(2x+1)=0$$

$\Rightarrow$ $\boxed{y=2x+1 \text{ is oblique asymptote.}}$

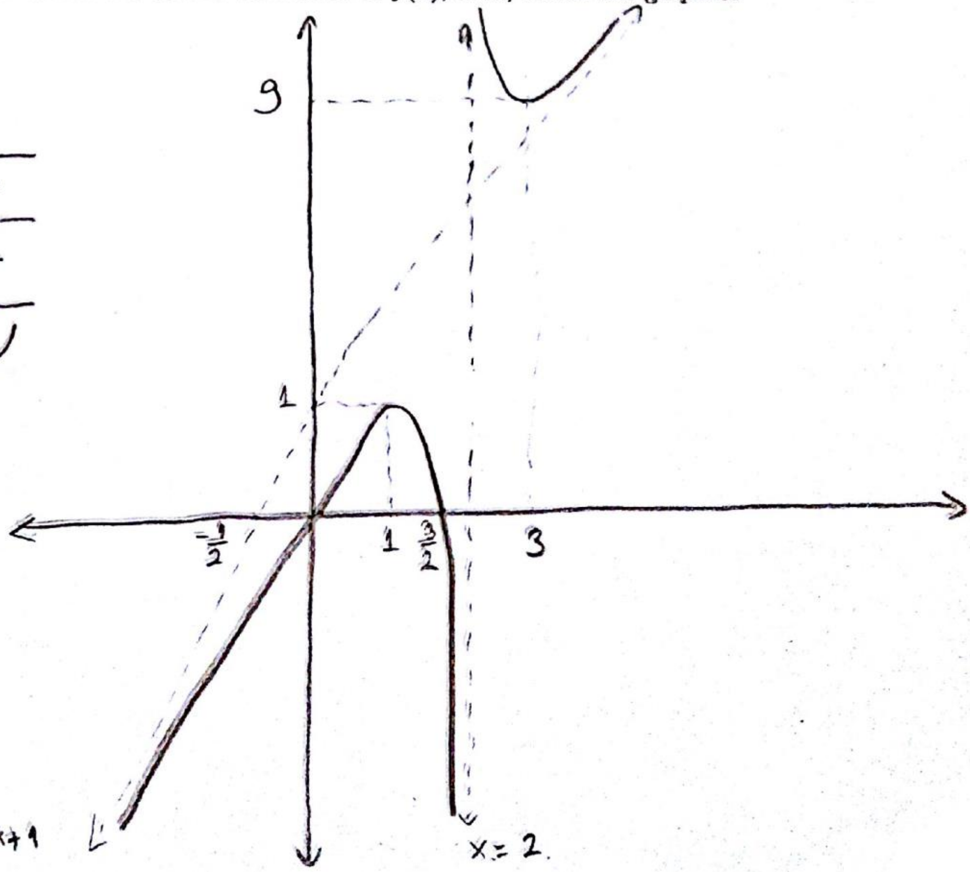
(b) Find the first and the second derivatives of $f(x)$.

$$f'(x)=\frac{(4x-3)(x-2)-(2x^2-3x)}{(x-2)^2}=\frac{2x^2-8x+6}{(x-2)^2}=\frac{2(x-3)(x-1)}{(x-2)^2}$$

$$f''(x)=\frac{(4x-8)(x-2)^2-2(x-2)(2x^2-8x+6)}{(x-2)^4}=\frac{(4x-8)(x-2)-2(2x^2-8x+6)}{(x-2)^3}=\frac{4}{(x-2)^3}$$

(c) Make a table of the signs of the first and the second derivatives of $f(x)$; next, sketch the graph of $f(x)$.

x		1		2		3	
f'	+	0	−	¦	−	0	+
f''	−		−	¦	+		+
f	⌒		⌒	¦	◡		◡



(10 pts) 2. Two positive numbers have product 49. What is the smallest possible value for their sum?

$xy = 49$ Minimize $f = x + y$ for $x, y > 0$

$\Rightarrow f(x) = x + \dfrac{49}{x}$ on $(0, \infty)$

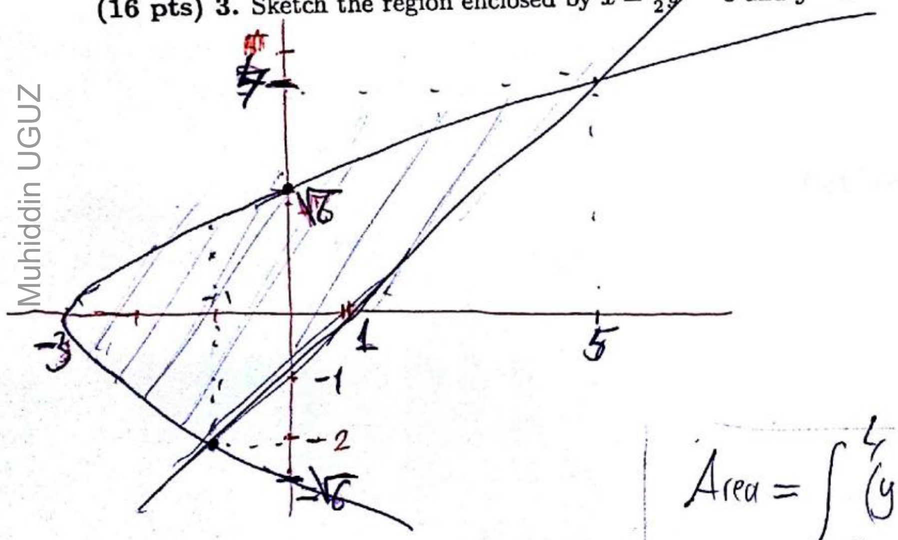
$f'(x) = 1 + \dfrac{-49}{x^2} = 0 \iff x = 7$ on $(0, \infty)$

f has local min at $x = 7$

x		7	
f'	$-$	0	$+$

1st DT for open intervals $\Rightarrow$ f has abs min at $x = 7$ which is $f(7) = 14$

(16 pts) 3. Sketch the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$ and determine its area.



Intersection points:

$\frac{1}{2}y^2 - 3 = y + 1$

$\Rightarrow (x, y) = (-1, -2)$

or $(x, y) = (5, 4)$

$$\text{Area} = \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy$$

$$= \int_{-2}^{4} \left(y + 1 - \tfrac{1}{2}y^2 + 3\right) dy = \frac{y^2}{2} + 4y - \frac{y^3}{6} \Big|_{-2}^{4}$$

$$= 18$$

Alternatively,

$$\text{Area} = \int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx$$

$$= \frac{2}{3}(2x+6)^{3/2} \Big|_{-3}^{-1} + \left(\frac{1}{3}(2x+6)^{3/2} - \frac{x^2}{2} + x\right) \Big|_{-1}^{5}$$

$$= 18$$

(8+8+8 pts) **4.** Evaluate the following indefinite integrals.

(a) $\int \sin^3 x \cos^4 x \, dx$ $= \int \sin^2 x \cdot \sin x \cdot \cos^4 x \, dx$

$= \int (1-\cos^2 x) \cos^4 x \cdot \sin x \, dx$ $\left(\text{Let } \cos x = u, \text{ then } -\sin x \, dx = du \right)$

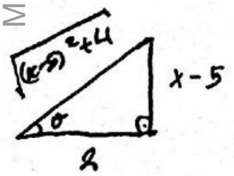
$= \int (u^2 - 1) u^4 \, du$

$= \frac{u^7}{7} - \frac{u^5}{5} + C$

$= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

(b) $\int \frac{x\,dx}{\sqrt{x^2 - 10x + 29}}$ $= \int \frac{x \, dx}{\sqrt{(x-5)^2 + 4}} = \frac{1}{2} \int \frac{x \, dx}{\sqrt{\left(\frac{x-5}{2}\right)^2 + 1}}$

$\left[\text{let } \frac{x-5}{2} = \tan\theta, \text{ then } \frac{dx}{2} = \sec^2\theta \, d\theta \right]$ $0 < \theta < \pi/2$



$= \frac{1}{2} \int \frac{(2\tan\theta + 5)\, 2\sec^2\theta \, d\theta}{\sec\theta}$

$= \int 2\tan\theta \sec\theta \, d\theta + \int 5 \sec\theta \, d\theta$

$= 2\sec\theta + 5 \ln|\sec\theta + \tan\theta| + C$

$= \sqrt{(x-5)^2+4} + 5 \ln\left| \frac{\sqrt{(x-5)^2+4}}{2} + \frac{x-5}{2} \right| + C$

(c) $\int x^3 \cos(x^2) \, dx = I$

$\left[\text{let } x^2 = t, \text{ then } 2x\,dx = dt \right]$

$I = \int x x^2 \cos(x^2) dx = \frac{1}{2} \int t \cos(t) \, dt$

by IBP

$= \frac{1}{2} \left[t \sin t - \int \sin(t) \, dt \right]$

IBP
let $t = u$, $\cos(t) dt = dv$
then $dt = du$, $\sin(t) = v$

$= \frac{1}{2} \left[t \sin t + \cos(t) \right] + C$

$= \frac{1}{2} \left[x^2 \sin(x^2) + \cos(x^2) \right] + C$

(26 pts) 5. Evaluate the integral $\int \frac{x^4+x^3+2x^2+2x+1}{x(x^2+1)^2}\,dx$.

$$I = \int \frac{x^4+x^3+2x^2+2x+1}{x.(x^2+1)^2}\,dx$$

$$\underset{(x^2+1)^2}{\frac{A}{x}} + \underset{(x)}{\frac{Bx^3+Cx^2+Dx+E}{(x^2+1)^2}} = \frac{x^4+x^3+2x^2+2x+1}{x.(x^2+1)^2}$$

$\Downarrow$

$A=1$
$B=0$
$C=1$
$D=0$
$E=2$

$$\Rightarrow I = \int \frac{dx}{x} + \int \frac{x^2+2}{(x^2+1)^2}\,dx = \underbrace{\int \frac{dx}{x}}_{\ln|x|} + \underbrace{\int \frac{dx}{1+x^2}}_{\arctan x} + \int \frac{dx}{(1+x^2)^2}$$

$$= \ln|x| + \arctan x + \int \frac{dx}{(1+x^2)^2} \overset{\text{by}①}{=} \ln|x| + \arctan x + \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \ln|x| + \arctan x + \int \cos^2\theta\,d\theta$$

$$\overset{\text{by}②}{=} \ln|x| + \arctan x + \int \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= \ln|x| + \arctan x + \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C$$

$$= \ln|x| + \arctan x + \frac{2}{4}\cdot\sin\theta.\cos\theta + \frac{\theta}{2} + C$$

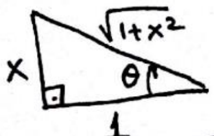
$$= \ln|x| + \arctan x + \frac{1}{2}\cdot\frac{x}{(1+x^2)} + \frac{1}{2}.\arctan x + C,$$

$C \in \mathbb{R}$

①

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$



$$\boxed{\cos^2\theta = \frac{\cos 2\theta + 1}{2}}\ ②$$

M E T U Department of Mathematics

	Math 119 Calculus With Analytic Geometry Final Exam	
Acad. Year : *2015-2016* Semester : *Spring* Coordinator: *Emre Coşkun* Date : *June 02, 2016* Time : *13:30* Duration : *105 minutes*	Last Name : Name : Department : Signature : Student No. : Section :	
	5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	SHOW YOUR WORK	

(8+8+8 pts) 1. Calculate the following limits:

(a) $\lim_{x\to1} \frac{(\ln(x))^2}{\ln(x^2)}$ $\left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{x\to1} \frac{2\ln x \cdot \frac{1}{x}}{\frac{1}{x^2}\cdot 2x} = \lim_{x\to1} \ln x = 0$$

$\underline{\underline{OR}}$ $$\lim_{x\to1} \frac{(\ln x)^2}{\ln(x^2)} = \lim_{x\to1} \frac{(\ln x)^2}{2\ln x} = \lim_{x\to1} \frac{\ln x}{2} = 0$$

(b) $\lim_{x\to\infty} (xe^{\frac{1}{x}} - x)$ $(\infty - \infty)$

$$\lim_{x\to\infty} x\cdot(e^{1/x}-1) = \lim_{x\to\infty} \frac{e^{1/x}-1}{\frac{1}{x}} \overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^{1/x}\cdot\left(\frac{-1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = 1.$$

(c) $\lim_{x\to0} (\cos x)^{\frac{1}{x^2}} = \lim_{x\to0^+} e^{\ln(\cos x)^{1/x^2}} = \lim_{x\to0^+} e^{\frac{\ln(\cos x)}{x^2}}$

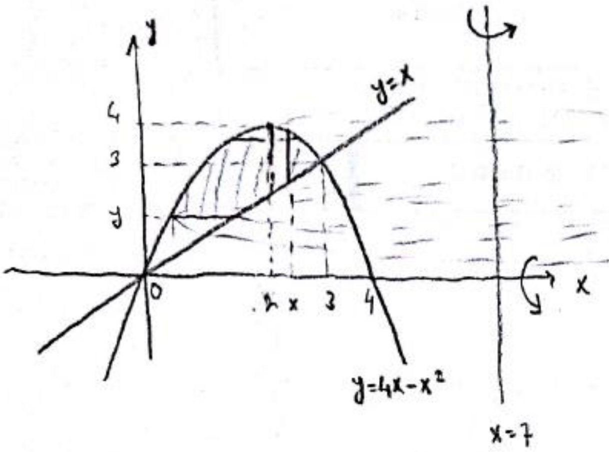
(since e^x is continuous)

$$= e^{\lim_{x\to0^+} \frac{\ln(\cos x)}{x^2}} \overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}} e^{\lim_{x\to0^+} \frac{\frac{-\sin x}{\cos x}}{2x}} \overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}} e^{\lim_{x\to0^+} \frac{-\sec^2 x}{2}}$$

$$= e^{-\frac{1}{2}}$$

(14+12 pts) **2.** Let R be the region bounded by the curves $y = x$ and $y = 4x - x^2$ in the xy-plane. Write the integrals which give the volumes of the solids described below. Do not evaluate the integrals.

(a) The solid obtained by rotating the region R about the line $x = 7$.



Using shell-method:

$$V = \int_0^3 2\pi(7-x)(4x-x^2-x)\,dx$$

Using Disc-method:

$$V = \pi\int_0^3 \left(\left[7-(2-\sqrt{4-y})\right]^2 - (7-y)^2\right)dy + \pi\int_3^4 \left[\left(7-(2-\sqrt{4-y})\right)^2 - \left(7-(2+\sqrt{4-y})\right)^2\right]dy$$

(b) The solid obtained by rotating the region R about the line $y = 0$.

$$V = \pi\int_0^3 \left((4x-x^2)^2 - x^2\right)dx$$

or

$$V = \int_0^3 2\pi y\left[y-(2-\sqrt{4-y})\right]dy + \int_3^4 2\pi y\left[(3+\sqrt{4-y}) - (2-\sqrt{4-y})\right]dy$$

(8+8+8 pts) 3. Evaluate the given improper integral or show that it is divergent:

(a) $\int_1^\infty \frac{1}{(3x+1)^2}dx = \lim_{R\to\infty} \int_1^R \frac{1}{(3x+1)^2}dx$ (let $3x+1=u$, then $3dx=du$)

$= \lim_{R\to\infty} -\frac{1}{3}\frac{1}{(3x+1)}\Big|_1^R$

$= \lim_{R\to\infty}\left[-\frac{1}{3(3R+1)} + \frac{1}{12}\right] = \frac{1}{12}$

(b) $\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \underbrace{\frac{1}{(x-2)(x+3)}}_{f(x)}dx$, integral is improper at $x=2$.

$I = \int_0^2 f(x)dx + \int_2^4 f(x)dx$ (if both converge)

Consider $\int_2^4 f(x)dx$, choose $g(x)=\frac{1}{x-2}$ where $g(x), f(x) > 0$ on $(2,4]$

$\lim_{x\to 2^+} \frac{f(x)}{g(x)} = \lim_{x\to 2^+}\frac{1}{x+3} = 5 > 0$. Since $\int_2^4 \frac{1}{x-2}dx = \int_0^2 \frac{1}{u}du$ is divergent (by p-test)

It follows that $\int_2^4 f(x)dx$ is also divergent by LCT.

Therefore, $\int_0^4 f(x)dx$ diverges //

(c) $\int_1^\infty \frac{1+e^{-2x}}{x}dx$

Observe that $0 \le \overbrace{\frac{1}{x}}^{f(x)} \le \overbrace{\frac{1+e^{-2x}}{x}}^{g(x)}$ where both $f(x)$ and $g(x)$ are positive and cont. on $[1,\infty)$

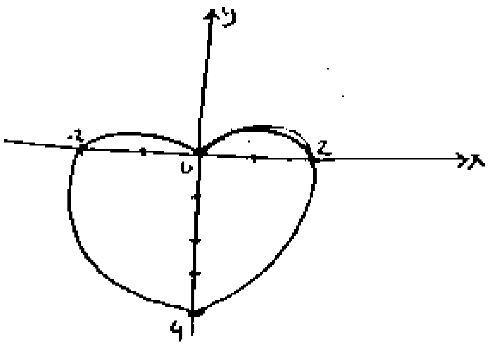
Since $\int_1^\infty \frac{1}{x}dx$ is divergent by p-test,

$\int_1^\infty g(x)dx$ is also divergent by comparison test

$\square$

(10 pts) 4. Sketch the graph of the polar curve $r = 2(1 - \sin\theta)$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0	$2-\sqrt{3}$	$2-\sqrt{2}$	1	2	3	$\sqrt{3}+2$	4	$2+\sqrt{3}$	3	2



OR you can show that the curve is symmetric about y-axis. So, it is enough to show the values for θ between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

(16 pts) 5. Find the length of the curve $y = \ln(\cos x), 0 \le x \le \frac{\pi}{3}$.

$$S = \int_0^{\pi/3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/3} \sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\pi/3} \frac{1}{\cos x}\,dx = \int_0^{\pi/3} \sec x\,dx = \ln|\sec x + \tan x|\Big|_{x=0}^{\pi/3}$$

$$\left\{ \frac{dy}{dx} = \frac{-\sin x}{\cos x} \right\} \qquad \left\{ \begin{array}{l} |\cos x| = \cos x \\ \text{for } 0 < x \le \frac{\pi}{3} \end{array} \right\}$$

$$= \ln|2+\sqrt{3}|$$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	List No.

Code : *Math 119* Acad. Year : *2011-2012* Semester : *Fall* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *November.19.2011* Time : *9:30* Duration : *119 minutes*	8 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (18 pts) Evaluate the following derivatives. There is **no** need to simplify your answer.

a) $\dfrac{d}{dx}(1+x)^{\sin x}$

Let $y = (1+x)^{\sin x}$. Then $\ln y = \sin x \ln(1+x)$. Taking derivative of both sides, we obtain $\dfrac{y'}{y} = \cos x . \ln(1+x) + \sin x . \dfrac{1}{1+x}$

Hence $y' = (1+x)^{\sin x}\left[\cos x . \ln(1+x) + \dfrac{\sin x}{1+x}\right]$

b) $\dfrac{d}{dx}\left(x^3 \arcsin(x^2 + 1)\right) = 3x^2 \arcsin(x^2-1) + x^3 \dfrac{2x}{\sqrt{1-(x^2-1)^2}}$

Note that in the exam x^2-1 is misprinted as x^2+1.
in that case; since $\arcsin x : [-1,+1] \longrightarrow [-\pi/2, \pi/2]$, domain of $x^3 \arcsin(x^2+1)$ consists of only one point, namely $x=0$; and hence is not differentiable.

BUT :
- if you realized this point and wrote that "derivative does not exist", you'll get full credit.
- if you did not realize it and applied product and chain rules as it is, you'll get full credit.

c) $\dfrac{d}{dx}f(x)$ at the point $(x, f(x)) = (1,1)$ where f is a differentiable function satisfying the equation $e^{xf(x)} + \ln f(x) = e$ for all x.

Take derivative of both sides: $e^{xf(x)}\left(f(x) + x f'(x)\right) + \dfrac{f'(x)}{f(x)} = 0$

substitude $\left.\begin{array}{l} x=1 \\ f(1)=1 \end{array}\right\}$ $e(1 + f'(1)) + f'(1) = 0$

Hence

$f'(1) = \dfrac{-e}{1+e}$

Question 2 (18 pts) Evaluate the following limits. If the limit does not exist, is it ∞, $-\infty$ or neither? Explain. (**Do not use l'Hôpital's Rule**).

a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x}\right) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^2+x}-\sqrt{x}\right)\left(\sqrt{x^2+x}+\sqrt{x}\right)}{\left(\sqrt{x^2+x}+\sqrt{x}\right)}$

$= \lim\limits_{x\to\infty} \dfrac{x^2+\cancel{x}-\cancel{x}}{\underbrace{|x|}_{=x}\sqrt{1+\frac{1}{x}}+x\frac{1}{\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{1+\frac{1}{x}}+\frac{1}{\sqrt{x}}} = \infty$

where $\frac{1}{x}\to 0$, $\frac{1}{\sqrt{x}}\to 0$, $\sqrt{1+\frac{1}{x}}+\frac{1}{\sqrt{x}}\to 1$

b) $\lim\limits_{x\to\infty}\dfrac{2x^2+\sin^2 x}{x^2} = \lim\limits_{x\to\infty} \dfrac{2x^2}{x^2}+\left(\dfrac{\sin x}{x}\right)^2 = \lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty}\left(\dfrac{\sin x}{x}\right)^2$

we can seperate the limit since each of the following limits exists

x^2 is a continuous function

$= 2 + \left(\lim\limits_{x\to\infty}\dfrac{\sin x}{x}\right)^2 = 2+0^2 = 2$

c) $\lim\limits_{x\to 0}\dfrac{|\sin(x)|}{x}$

$$\left.\begin{aligned}\lim_{x\to 0^-}\frac{|\sin x|}{x} &= \lim_{x\to 0^-}\frac{-\sin x}{x} = -1 \\ &\ne \\ \lim_{x\to 0^+}\frac{|\sin x|}{x} &= \lim_{x\to 0^+}\frac{\sin x}{x} = 1\end{aligned}\right\} \Rightarrow \lim_{x\to 0}\frac{|\sin x|}{x} \text{ does not exist}$$

Question 3 (10 pts) Let $f : \mathbf{R}\to\mathbf{R}$ be a function defined by

$$f(x) = \begin{cases} mx - b^2 & \text{if} \quad x < 119 \\ x^2 & \text{if} \quad x \ge 119 \end{cases}$$

Using the definition of derivative, find all values of m and b that make f differentiable at 119.

since f is going to be differentiable at 119, we must have

$$\lim_{t\to 0^+}\frac{f(119+t)-f(119)}{t} = \lim_{t\to 0^-}\frac{f(119+t)-f(119)}{t}$$

$\Rightarrow \lim\limits_{t\to 0^+}\dfrac{(119+t)^2-119^2}{t} = \lim\limits_{t\to 0^+}\dfrac{2\cdot 119\cancel{t}+t\cancel{^2}}{\cancel{t}} = 2\cdot 119 = \lim\limits_{t\to 0^-}\dfrac{m(119+t)-b^2-119^2}{t}$

$\Rightarrow \lim\limits_{t\to 0^-}\left[\dfrac{m\cdot 119-b^2-119^2}{t}+m\right] = 2\cdot 119$. But this limit exists if and only if

$m\cdot 119 - b^2 - 119^2 = 0$ and in this case $m = 2\cdot 119$

Hence $\cancel{2}\cdot 119^2 - b^2 - \cancel{119^2} = 0 \Rightarrow b = \mp 119$

Question 4 (9 pts) Prove that $\ln(1+x^2) \leq 2x$ for every real number $x \in (0,1)$.

Let $f(x) = \ln(1+x^2)$ which is continuous on $[0,1]$ and differentiable on $(0,1)$

also note that $f(0) = \ln 1 = 0$.

For any $x \in (0,1)$, since $[0,x] \subseteq [0,1]$, f is continuous on $[0,x]$ and differentiable on $(0,x)$ hence

we can apply Mean Value Theorem to f on $[0,x]$:

$$\frac{f(x)-f(0)}{x-0} = f'(c) \quad \text{for some } c \in (0,x) \subseteq (0,1)$$

That is

$$\frac{\ln(1+x^2)}{x} = \frac{2c}{1+c^2} = 2 \cdot \underbrace{\frac{c}{1+c^2}}_{\leq 1 \text{ since } c \in (0,1)} \leq 2$$

For an alternative solution see page 5.

Thus $\ln(1+x^2) \leq 2x \quad \forall x \in (0,1)$

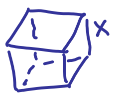
Question 5 (12 pts) Using the definition of limit prove that $\lim\limits_{x\to 2}(3x-1) = 5$.

Given $\varepsilon > 0$, take $\delta = \varepsilon/3 > 0$.

If $0 \neq |x-2| < \delta$, we have

$$|(3x-1)-5| = |3x-6| = 3|x-2| < 3\delta = \varepsilon$$

Question 6 (9 pts) How fast is the surface area of a cube changing when the volume of the cube is 64 cm^3 and is increasing at 2 cm^3/sec?



Surface area $= S(x) = 6x^2$

volume $= V(x) = x^3$

Dimensions are changing : so in fact $x = x(t)$ function of time

and we are given that, at the moment $t = t_0$ we have

$V(x(t_0)) = 64 \text{ cm}^3 \Rightarrow x(t_0) = 4$ cm.

also

$$\left.\frac{dV}{dt}\right|_{t=t_0} = +2 \text{ cm}^3/\text{sec} = 3x^2(t_0) \cdot \left.\frac{dx}{dt}\right|_{t=t_0} = 3\cdot 16 \cdot \left.\frac{dx}{dt}\right|_{t=t_0}$$

$$\Rightarrow \left.\frac{dx}{dt}\right|_{t=t_0} = \frac{2}{3\cdot 16} = \frac{1}{24} \text{ cm}$$

Thus

$$\left.\frac{dS}{dt}\right|_{t=t_0} = 12\,x(t_0) \cdot \left.\frac{dx}{dt}\right|_{t=t_0} = 12 \cdot 4 \cdot \frac{1}{24} = 2 \text{ cm}^2/\text{sec}$$

Question 7 (12 pts) Let $f(x) = 2x + \cos(x)$.

a) Show that f is invertible.

$f'(x) = 2 - \sin x \cancel{\geq} > 0 \ \forall x \Rightarrow f(x)$ is strictly increasing on $\mathbb{R}$

$\Rightarrow f(x)$ is one to one on $\mathbb{R}$

$\Rightarrow f(x)$ is invertible

b) Find $(f^{-1})(\pi)$ and the **derivative of the inverse** of f at π, that is, find $(f^{-1})'(\pi)$.

since $f(\pi/2) = 2\cdot\frac{\pi}{2} + \cos\frac{\pi}{2} = \pi$, we have $f^{-1}(\pi) = \pi/2$

since $(f^{-1})'(\pi) = \dfrac{1}{f'(f^{-1}(\pi))} = \dfrac{1}{f'(\pi/2)} = \dfrac{1}{2-\sin \pi/2} = \dfrac{1}{2-1} = 1$

Muhiddin UGUZ

Question 8 (12 pts) Prove that the tangent lines drawn to the ellipse $x^2 + 2y^2 = 2$ and to the hyperbola $2x^2 - 2y^2 = 1$ (at the points of intersections) intersect at right angles.

First, lets find the intersection points of given two curves;

$x^2 + 2y^2 = 2 \Rightarrow 2y^2 = 2 - x^2$. substituting this in the second equation;

$2x^2 + x^2 - 2 = 1 \Rightarrow 3x^2 = 3 \Rightarrow x = \mp 1 \Rightarrow y = \mp \frac{1}{\sqrt{2}}$

$\Rightarrow$ They intersect at 4 points.

$x^2 + 2y^2 = 2 \Rightarrow \cancel{2}x + \cancel{4}^{2} y y' = 0 \Rightarrow y' = \dfrac{-x}{2y}$

$2x^2 - 2y^2 = 1 \Rightarrow \cancel{4}x - \cancel{4} y y' = 0 \Rightarrow y' = \dfrac{x}{y}$

we need to show $\dfrac{-x}{2y} \cdot \dfrac{x}{y} = -1$ at the intersection points.

At the intersection points, that is when $(x,y) = (\mp 1, \mp \frac{1}{\sqrt{2}})$, we have

$$\frac{-x}{2y} \cdot \frac{x}{y} = \frac{-1}{2}\frac{x^2}{y^2} = \frac{-1}{2}\left(\frac{\mp 1}{\mp \frac{1}{\sqrt{2}}}\right) = \frac{-1}{2}\,\frac{1}{\frac{1}{2}} = -1$$

An alternative solution to question 4

To prove $\ln(1+x^2) \le 2x \quad \forall x \in (0,1)$;

Define $g(x) = \ln(1+x^2) - 2x$ which is differentiable on $(0,1)$

since $g(0) = 0$, it is enough to show $g(x)$ is strictly decreasing on $(0,1)$.

since $g'(x) = \frac{2x}{1+x^2} - 2 = \frac{2x-2-2x^2}{1+x^2} = \frac{-2}{1+x^2}\underbrace{(x^2-x+1)}_{\text{has no solution hence always } \oplus}$

$\frac{-2}{1+x^2}$ always $\ominus$

$\Rightarrow g'(x) < 0 \quad \forall x \in (0,1)$

$\Rightarrow g(x) \searrow \quad \Rightarrow g(x) \le g(0) = 0$

$\Rightarrow \ln(1+x^2) \le 2x$

An alternative solution to question 7/a

To prove $f(x) = 2x + \cos x$ is one to one

Assume $f(a) = f(b)$ and $a \lneq b$

Then

$$2a + \cos a = 2b + \cos b$$

$$\Rightarrow 2(a-b) = \cos b - \cos a$$

$$\Rightarrow \frac{\cos b - \cos a}{b-a} = -2$$

But $\cos x$ is continuous and differentiable everywhere

$$\xRightarrow{\text{Mean Value Thm}} -\sin c = -2 \quad \text{for some } c \in (a,b)$$

$$\Rightarrow \sin c = 2 \quad \text{contradiction.}$$

Hence there is NO such $a \lneq b$. $\Rightarrow f$ is one to one

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Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY Final Exam	List No.

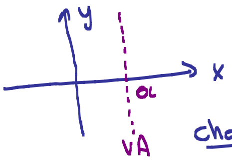
Code : *Math 119* Acad. Year : *2011-2012* Semester : *Fall* Coordinator: *Muhiddin Uğuz* Date : *January.12.2012* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

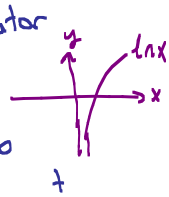
Question 1 (18 pts) Let $f(x) = \dfrac{\ln x}{x^2}$.

a) Find all vertical and horizontal asymptotes of the graph of f, if there are any.

• Vertical Asymptotes :



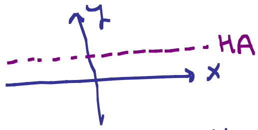
candidates are zeros of denominator so $x=0$ is a candidate



check $\lim_{x\to 0^+} \dfrac{\ln x}{x^2} = \lim_{x\to 0} \underbrace{\dfrac{1}{x^2}}_{+\infty} \cdot \underbrace{\ln x}_{-\infty} = -\infty$

Hence $x=0$ is V.A.

• Horizontal Asymptotes:



$\lim_{x\to+\infty} \dfrac{\ln x}{x^2}$ $\left(\dfrac{\infty}{\infty}\text{ type}\right)$ $\underset{\text{L'Hopital's Rule}}{=} \lim_{x\to+\infty} \dfrac{1/x}{2x} = \lim_{x\to+\infty} \dfrac{1}{2x^2} = 0 < \infty$

Hence $y=0$ is H.A

b) Find all intervals where f is increasing and all intervals where f is decreasing.

$f(x) = \dfrac{\ln x}{x^2} \Rightarrow f'(x) = \dfrac{\frac{1}{x}x^2 - 2x\ln x}{x^{4}}= \dfrac{1-2\ln x}{x^3} = 0 \Rightarrow \ln x = \frac{1}{2}$

$x^3 > 0$ on Domain $=(0,+\infty)$ $\Rightarrow x = e^{1/2} = \sqrt{e}$

x		0		$\sqrt{e}$	
f'	//////		+	0	−
f	//////		↗		↘

f is increasing on $(0,\sqrt{e})$
decreasing on $(\sqrt{e},+\infty)$

c) Find all intervals where f is concave up and all intervals where f is concave down.

$f''(x) = \dfrac{-\frac{2}{x}x^{3} - 3x^{2}(1-2\ln x)}{x^{6}} = \dfrac{-2-3+6\ln x}{x^4} = \dfrac{6\ln x - 5}{x^4} = 0 \Rightarrow \ln x = 5/6$

$x^4 > 0$ $\Rightarrow x = e^{5/6}$

x		0		$e^{5/6}$	
f''	//////		−	0	+
f	//////		∩		∪

f is concave up on $(e^{5/6},+\infty)$
concave down on $(0, e^{5/6})$

Question 2 (18 pts)

a) Evaluate $\lim_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) = \lim_{x\to 1} \dfrac{x\ln x - x + 1}{(x-1)\ln x}$ $\left(\frac{0}{0}\text{ type}\right)$

$$\overset{\text{L'Hop's Rule}}{=} \lim_{x\to 1} \frac{\ln x + \cancel{1} - \cancel{1}}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} \left(\frac{0}{0}\text{ type}\right)$$

$$= \lim_{x\to 1} \frac{\cancel{1/x}}{\cancel{1/x} + 1/x^2} = \lim_{x\to 1} \frac{1}{1+\frac{1}{x}} = 1/2$$

b) Find $\dfrac{d}{dx}\left[\dfrac{1}{\ln(1+2^x)}\right] = \dfrac{-\frac{2^x \ln 2}{1+2^x}}{\ln^2(1+2^x)}$

$$= -\frac{2^x \ln 2}{(1+2^x)\ln^2(1+2^x)}$$

$y = 2^x \Rightarrow \ln y = x\ln 2$

$\Rightarrow \dfrac{y'}{y} = \ln 2$

$\Rightarrow y' = 2^x \cdot \ln 2$

c) Evaluate $\dfrac{d}{dx}\left[\underbrace{\dfrac{(1+x)e^x}{(1-x)^2(3-x)^3}}_{\overset{\shortparallel}{y(x)}}\right]$ at $x = 0$

$y(x) \rightarrow y(0) = \frac{1}{27}$

Use logarithmic differentiation;

$$\ln y(x) = \ln(1+x) + x - 2\ln(1-x) - 3\ln(3-x)$$
$$= \ln(1+x) + x - 2\ln(1-x) - 3\ln(3-x)$$

Taking derivative of both sides, we obtain;

$$\frac{y'(x)}{y(x)} = \frac{1}{1+x} + 1 + \frac{2}{1-x} + \frac{3}{3-x} \quad \underset{x=0}{\overset{}{\xRightarrow{\text{at}}}} \quad y'(0) = \frac{1}{27}(1+1+2+1)$$

$$= 5/27$$

Question 3 (18 pts) Evaluate the following

a) $\displaystyle\int \frac{x^3+1}{x^3-x^2}\,dx = \int 1 + \frac{x^2+1}{x^3-x^2}\,dx = \int 1 + \frac{\cancel{x^2}}{\cancel{x^2}(x-1)} + \frac{1}{x^2(x-1)}\,dx$

$$\begin{array}{r|l} x^3+1 & x^3-x^2 \\ \hline + \; -x^3+x^2 & 1 \\ \hline x^2+1 & \end{array}$$

$$= x + \ln|x-1| + \int \frac{A}{\underset{x(x-1)}{x}} + \frac{B}{\underset{(x-1)}{x^2}} + \frac{C}{\underset{(x^2)}{x-1}}\,dx$$

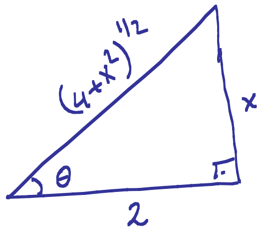
$$Ax^2 - Ax + Bx - B + Cx^2 = 1$$

$$x^2\underbrace{(A+C)}_{=0} + x\underbrace{(B-A)}_{=0}\underbrace{-B}_{=1} = 1 \quad \Big\} \quad B=-1 \Rightarrow A=-1 \Rightarrow C=+1$$

$$= x + \ln|x-1| + \int \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\,dx$$

$$= x + 2\ln|x-1| - \ln|x| + \frac{1}{x} + C = x + \frac{1}{x} + \ln\frac{(x-1)^2}{x} + c$$

b) $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{1}{(4+x^2)^{3/2}}\,dx = \int \frac{1}{8}\cos^{3}\theta \cdot 2\cancel{\sec^2\theta}\,d\theta$



$$\frac{1}{2}\cos\theta = \frac{1}{(4+x^2)^{1/2}}$$

$$2\tan\theta = x$$

$$2\sec^2\theta\,d\theta = dx$$

$$= \frac{1}{4}\int \cos\theta\,d\theta$$

$$= \frac{1}{4}\sin\theta + C$$

$$= \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + C$$

c) $\displaystyle\int e^{\sqrt[3]{x}}\,dx = \int e^y\,3y^2dy = 3\int \underbrace{y^2}_{u}\,\underbrace{e^y\,dy}_{dv} = 3\Big[y^2e^y - 2\int \underbrace{y}_{u}\,\underbrace{e^y dy}_{dv}\Big]$

$du = 2y\,dy$
$v = e^y$

$du = dy$
$v = e^y$

$$y = \sqrt[3]{x} = x^{1/3}$$

$$dy = \frac{1}{3}x^{-2/3}dx$$

$$\Rightarrow dx = 3x^{2/3}dy = 3y^2\,dy$$

$$= 3\Big[y^2e^y - 2\big(ye^y - \int e^y dy\big)\Big]$$

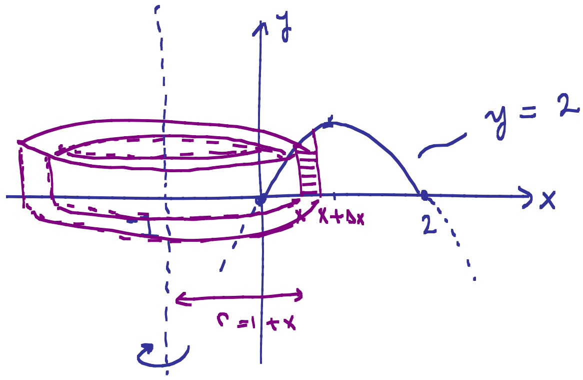
$$= 3y^2e^y - 6ye^y + 2e^y + C$$

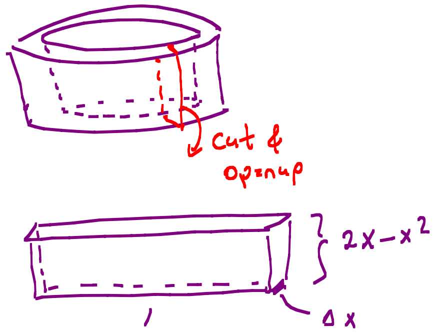
$$= e^y(3y^2 - 6y + 6) + C$$

$$= e^{\sqrt[3]{x}}(3x^{2/3} - 6x^{1/3} + 6) + C$$

Question 4 (14 pts) Let R be the region bounded from below by the x-axis and from above by the curve $y = 2x - x^2$. R is rotated about the line $x = -1$ to generate a solid. Write (**DO NOT EVALUATE**) integrals giving the volume V of the resulting solid of revolution by using following methods;

a) The cylindrical shell method. $\Rightarrow$ Take a slice parallel to rotation axis.



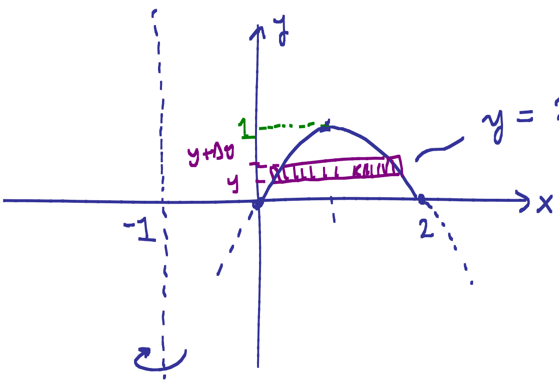


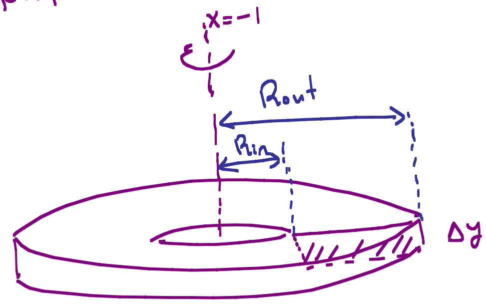
$2\pi r = 2\pi(1+x)$

$\Delta V_i = 2\pi(1+x)(2x-x^2)\Delta x_i$

$$V = \lim_{\|P\| \to 0} \sum \Delta V_i = 2\pi \int_0^2 (1+x)(2x-x^2)\,dx$$

b) The disc (slicing, washer) method. $\Rightarrow$ Take a slice perpendicular to rotation axis.





$y = 2x - x^2$

$x^2 - 2x + y = 0$

$x_{1,2} = \dfrac{2 \mp \sqrt{4-4y}}{2} = 1 \mp \sqrt{1-y}$

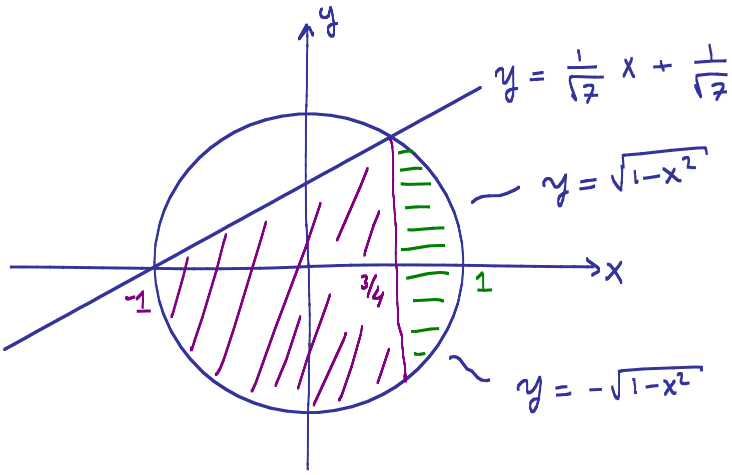
$$\Delta V_i = (R_{out}^2 - R_{in}^2)\pi \cdot \Delta y_i$$
$$= \left[(1 + \sqrt{1-y} + 1)^2 - (1 - \sqrt{1-y} + 1)^2\right] \pi \Delta y_i$$
$$= \left[(2 + \sqrt{1-y})^2 - (2 - \sqrt{1-y})^2\right] \pi \Delta y_i$$
$$= 8\sqrt{1-y}\, \pi \Delta y_i$$

$\Downarrow$

$$V = \lim_{\|P\| \to 0} \sum \Delta V_i = 8\pi \int_0^1 \sqrt{1-y}\, dy$$

NAME:................SURNAME:..................STUDENT ID:...........SIGNATURE:.........

Question 5 (14 pts) Write (**DO NOT EVALUATE**) an integral (or sum of integrals) which gives the area of the region lying below the line $\sqrt{7}y = x + 1$ and inside the circle $x^2 + y^2 = 1$.



$\sqrt{7}y = x + 1 \Rightarrow 7y^2 = (x+1)^2$

$7x^2 + (x+1)^2 = 7$

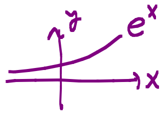
$8x^2 + 2x - 6 = 0$

$4x^2 + x - 3 = 0$

$x_1 = \frac{3}{4}$, $x_2 = -1$ are intersection points.

$$\text{Area} = \int_{-1}^{3/4} \left[\frac{1}{\sqrt{7}}(x+1) - \left(-\sqrt{1-x^2}\right)\right] dx + 2\int_{3/4}^{1} \sqrt{1-x^2}\, dx$$

Question 6 (18 pts) Determine whether following improper integrals converge or diverge. Give explanations.



a) $\displaystyle\int_{-\infty}^{0} xe^x dx = \lim_{c\to-\infty} \int_c^0 xe^x dx = \lim_{c\to-\infty} e^x(x-1)\Big|_c^0$

$$= \lim_{c\to-\infty} \left[(-1) - e^c (c-1)\right]$$

$$= -\lim_{c\to-\infty} 1 + \frac{c-1}{e^{-c}} \quad \left(\frac{\infty}{\infty} \text{ type}\right)$$

$$\underset{\text{L'Hopi Rule}}{=} -1 - \lim_{c\to-\infty} \frac{1}{-e^{-c}} = -1-0$$

$$= -1 \Rightarrow \text{convergent}$$

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = xe^x - \int e^x dx = xe^x - e^x + c = e^x(x-1)+c$$

$du = dx$

$v = e^x$

b) $\displaystyle\int_0^\infty \frac{dx}{x^{\frac{1}{2}}+x^2}$ This integral is improper at both $x=0$ and $x=\infty$

Consider $\displaystyle\int_0^{119} \frac{1}{x^{1/2}+x^2}dx$; since $x^{1/2}+x^2 \geq x^{1/2}$, we have

so we can use comparison test $\Leftarrow$ $0 \leq \dfrac{1}{x^{1/2}+x^2} \leq \dfrac{1}{x^{1/2}}$ $\forall x \in (0, 119)$

Hence

$$0 \leq \underbrace{\int_0^{119} \frac{1}{x^{1/2}+x^2}dx}_{\text{convergent by comparison test}} \leq \underbrace{\int_0^{119} \frac{1}{x^{1/2}}dx}_{\text{convergent by p-test}}$$

Now consider $\displaystyle\int_{119}^{\infty} \frac{1}{x^{1/2}+x^2}dx$; $\to 0 \Rightarrow$ we can use LCT.

since $\displaystyle\lim_{x\to\infty} \frac{\frac{1}{x^{1/2}+x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2+x^{1/2}} = 1$ a nonzero finite number,

we know that either both $\displaystyle\int_{119}^{\infty} \frac{1}{x^{1/2}+x^2}dx$ and $\displaystyle\int_{119}^{\infty} \frac{1}{x^2}dx$ converge

or both diverge.

Since $\displaystyle\int_{119}^{\infty} \frac{1}{x^2}dx$ is convergent by p-test, we have $\displaystyle\int_{119}^{\infty} \frac{1}{x^{1/2}+x^2}dx$ is convergent

Thus

being sum of two convergent improper integrals,

$\displaystyle\int_0^\infty \frac{1}{x^{1/2}+x^2}dx$ is also convergent

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	List No.

Code : *Math 119* Acad. Year : *2010-2011* Semester : *Fall* Coordinator : *Muhiddin Uğuz* Date : *November.13.2010* Time : *9:30* Duration : *119 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Notation: $\arcsin(x), \arccos(x), \arctan(x)$ denote inverse functions of the corresponding trigonometric functions. That is, $\arcsin(x) = \sin^{-1}(x)$.

Question 1 (18 pts) Evaluate the following derivatives. There is **no** need to simplify your answer.

a) $\dfrac{d}{dx}[\arcsin(e^x + x^e)]$

$y = \arcsin(f(x)) \Rightarrow \sin y = f(x)$
$\Rightarrow y' \cos y = f'(x)$
$\Rightarrow y' = \dfrac{f'(x)}{\cos x} = \dfrac{f'(x)}{\sqrt{1-\sin^2 x}}$
$\Rightarrow y' = \dfrac{f'(x)}{\sqrt{1-f^2(x)}}$

$$y' = \frac{e^x + e x^{e-1}}{\sqrt{1-(e^x + x^e)^2}}$$

b) $\dfrac{d}{dx}[(\sin x)^{\cos x}]$

$y = (\sin x)^{\cos x} \Rightarrow \ln y = \cos x . \ln(\sin x)$

$\Rightarrow \dfrac{y'}{y} = (-\sin x)\ln(\sin x) + \cos x . \dfrac{\cos x}{\sin x}$

$$y' = (\sin x)^{\cos x}\left[-(\sin x)\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$$

c) $\dfrac{d}{dx}[\tan^2(\sec(x^2))] = \left(2\tan(\sec(x^2))\right) . \left(\sec^2(\sec(x^2))\right) . \left(\sec(x^2)\tan(x^2)\right) . 2x$

$$\left[\begin{array}{l} (\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos^2 x - (-\sin x)\sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x \\ (\tan f(x))' = f'(x) \sec^2(f(x)) \end{array}\right]$$

$$\left[\begin{array}{l} (\sec x)' = \left(\dfrac{1}{\cos x}\right)' = \dfrac{0-(-\sin x)}{\cos^2 x} = \dfrac{1}{\cos x} . \dfrac{\sin x}{\cos x} = \sec x . \tan x \\ (\sec f(x))' = f'(x) . \sec f(x) . \tan f(x) \end{array}\right.$$

Question 2 (30 pts) Evaluate the following limits. If the limit does not exist, is it ∞, $-\infty$ or neither? (Do **not** use l'Hôpital's Rule).

a) $\displaystyle\lim_{x\to 1}\frac{x^2-2x+1}{x-1} = \lim_{\substack{x\to 1\\ (x\neq 1)}} \frac{(x-1)^{\cancel{2}}}{\cancel{(x-1)}} = \lim_{x\to 1}(x-1) = 0$

b) $\displaystyle\lim_{x\to 0}\frac{x^3\sin\left(\frac{1}{x}\right)}{\sin(x)} = \lim_{x\to 0}\left[\underbrace{\frac{x}{\sin x}}_{\to 1}\ \underbrace{\left(x^2\sin\left(\frac{1}{x}\right)\right)}_{\to 0}\right]$

since $-1 \le \sin\frac{1}{x} \le 1$

we have $\underbrace{-x^2}_{\to 0} \le x^2\sin\frac{1}{x} \le \underbrace{x^2}_{\to 0}$

$= 0$

c) $\displaystyle\lim_{x\to\infty}\frac{2-x+\sin(x)}{x+\cos(x)} = \lim_{x\to\infty}\frac{\underbrace{\frac{2}{x}}_{\to 0} - \underbrace{\left(1+\underbrace{\frac{\sin x}{x}}_{\to 0}\right)}_{\to 1}}{\underbrace{1+\underbrace{\frac{\cos x}{x}}_{\to 0}}_{\to 1}} = \frac{-1}{1} = -1$

d) $\displaystyle\lim_{x\to -\infty}\left(\sqrt{2x^2-x+1}-\sqrt{x^2-x+5}\right) = \lim_{x\to-\infty}\frac{(\cancel{2}x^2\cancel{-x}+1)-(\cancel{x^2}\cancel{-x}+5)}{\sqrt{2x^2-x+1}+\sqrt{x^2-x+5}}$

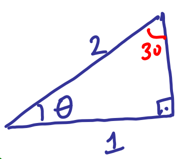
$\displaystyle= \lim_{x\to-\infty}\frac{\underbrace{x^{\cancel{2}}}_{\to -\infty}\underbrace{\left(1-\underbrace{\frac{4}{x^2}}_{\to 0}\right)}_{\to 1}}{|x|\left(\underbrace{\sqrt{2-\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{1}{x^2}}_{\to 0}}}_{\to\sqrt{2}}+\underbrace{\sqrt{1-\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{5}{x^2}}_{\to 0}}}_{\to 1}\right)} \to +\infty$

$|x| = -\cancel{x}$

e) $\displaystyle\lim_{x\to\frac{1}{2}}\frac{\arccos(x)-\frac{\pi}{3}}{x-\frac{1}{2}} = \lim_{x\to\frac{1}{2}}\frac{f(x)-f\left(\frac{1}{2}\right)}{x-\frac{1}{2}} = f'\left(\frac{1}{2}\right) = \frac{-1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{-1}{\sqrt{\frac{3}{4}}} = \frac{-2}{\sqrt{3}}$

let $f(x) = \arccos(x)$

then $f\left(\frac{1}{2}\right) = \arccos\left(\frac{1}{2}\right) = \theta = 60^\circ = \pi/3$



$\frac{4}{4}-\frac{1}{4} = \sqrt{\frac{3}{4}}$

$\frac{\sqrt{3}}{2}$

$y = \arccos x$

$\cos y = x$

$-y'\sin y = 1$

$y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

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Question 3 (18 pts) Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined as

$$f(x) = \begin{cases} x + x^2 & \text{if} \quad x < 1 \\ x^3 + 1 & \text{if} \quad x \geq 1 \end{cases}$$

a) Find the derivative $f'(x)$ for $x \neq 1$.

if $x < 1$, then $f(x) = x + x^2$ and hence $f'(x) = 1 + 2x$

if $x > 1$, then $f(x) = x^3 + 1$ and hence $f'(x) = 3x^2$

Therefore

$$f'(x) = \begin{cases} 1 + 2x & \text{if } x < 1 \\ 3x^2 & \text{if } x > 1 \end{cases}$$

b) Find $f'(1)$, if exists.

$$\lim_{x \to 1^+} \frac{f(x) - \overset{2}{\overset{\shortparallel}{f(1)}}}{x - 1} = \lim_{x \to 1^+} \frac{x^3 + 1 - 2}{x - 1} = \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}} = 3$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}} = 3$$

Thus $f'(1) = \lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = 3$ //

c) Is the **derivative** function $y = f'(x)$ continuous at $x = 1$? Explain.

from part (a), we have

$$\left.\begin{aligned} \lim_{x \to 1^-} f'(x) &= \lim_{x \to 1^-} 1 + 2x = 3 \\ \lim_{x \to 1^+} f'(x) &= \lim_{x \to 1^+} 3x^2 = 3 \end{aligned}\right\} \lim_{x \to 1} f'(x) = 3 = f'(1)$$

from part (b)

$\therefore$ $f'(x)$ is continuous at $x = 1$

Question 4 (12 pts) Find an equation of the tangent line to the curve
$119\left(\dfrac{x}{y}\right) + 1891\left(\dfrac{y}{x}\right)^3 = 2010$ at the point $(-1, -1)$.

Take derivative of both sides:

$$119\left(\frac{y - xy'}{y^2}\right) + 1891 \cdot 3\left(\frac{y}{x}\right)^2 \cdot \left(\frac{y'x - y}{x^2}\right) = 0$$

when $(x, y) = (-1, -1)$ we have; (note that this point is on the given curve, since it satisfies the given equation)

$$119(-1 + y') + 1891 \cdot 3 \cdot (-y' + 1) = 0$$

$$(1 - y')(1891 \cdot 3 - 119) = 0 \Rightarrow (1 - y') = 0 \Rightarrow y'(-1,-1) = 1$$

Tangent line $y - \overset{-1}{y_0} = \underset{y'(-1,-1) = 1}{m}(x - \overset{-1}{x_0}) \Rightarrow y + 1 = x + 1$

$\Rightarrow y = x$ //

Question 5 (10 pts) Suppose that $4 \leq f'(x) \leq 6$ for all $x \in \mathbf{R}$ and $f(0) = 119$. Show that $121 \leq f(\frac{1}{2}) \leq 122$.

Since f is differentiable $\forall x \in \mathbb{R}$ we have, by the Mean Value Theorem,

$$\frac{f(\frac{1}{2}) - \overset{=119}{f(0)}}{\frac{1}{2} - 0} = \underset{\geq 4}{\overset{\leq 6}{f'(c)}} \quad \text{for some } c \in (0, 1/2)$$

Thus

$$4 \leq \left(f(\tfrac{1}{2}) - 119\right) \cdot 2 \leq 6$$

$$2 \leq f(\tfrac{1}{2}) - 119 \leq 3$$

$$\Rightarrow \quad 121 \leq f(\tfrac{1}{2}) \leq 122$$

Question 6 (12 pts) Let $f(x) = \arctan(x) + e^x$.

a) Show that f is invertible, that is, f^{-1} is a function.

$$f'(x) = \underbrace{\frac{1}{1+x^2}}_{>0} + \underbrace{e^x}_{>0} > 0;$$ hence $f(x)$ is strictly increasing $\forall x$ therefore $f(x)$ is one to one and hence invertable.

b) Find the **derivative of the inverse** of f at 1, that is, find $(f^{-1})'(1)$.

Since $(f^{-1} \circ f)(x) = x$, taking derivative of both sides,

we get $(f^{-1})'(f(x)) \cdot f'(x) = 1$

Hence $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$ (if $f'(a) \neq 0$)

Note that $f(0) = \arctan 0 + e^0 = 0 + 1 = 1$

hence $f^{-1}(1) = 0$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{\left(\frac{1}{1+x^2} + e^x\right)_{x=0}} = \frac{1}{1+1} = \frac{1}{2}$$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANAYTIC GEOMETRY MidTerm 2	List No.
Code : *Math 119* Acad. Year : *2010-2011* Semester : *Fall* Coordinator : *Muhiddin Uğuz* Date : *December.18.2010* Time : *9:30* Duration : *110 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 (+4 Bonus) POINTS	

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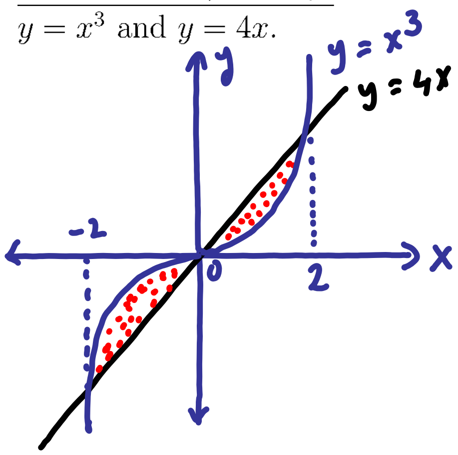
Question 1 (10 pts) Sketch and find the area of the finite region bounded by the curves $y = x^3$ and $y = 4x$.



$x^3 = 4x \Rightarrow x = 0, \mp 2$

$A = 2\int_0^2 4x - x^3 \, dx = 2\left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= 2\,[8 - 4] = 8 \text{ unit}^2$

Question 2 (16 pts) Evaluate the following indefinite integrals:

a) $\int \frac{\sin^3 x}{\cos^4 x} dx.$ $= -\int \frac{(1-u^2)}{u^4} du = \int u^{-2} - u^{-4} \, du = -\frac{1}{u} + \frac{1}{3u^3} + c$

Let $u = \cos x$

then $\frac{du}{dx} = -\sin x$

$= \frac{-1}{\cos x} + \frac{1}{3\cos^3 x} + c$

b) $\int (x+1)(x+2)^{119} dx.$ $= \int (u-1) u^{119} du = \int u^{120} - u^{119} du$

Let $u = x + 2$

then $\frac{du}{dx} = 1$

$= \frac{u^{121}}{121} - \frac{u^{120}}{120} + c$

$= \frac{(x+2)^{121}}{121} - \frac{(x+2)^{120}}{120} + c$

Question 3 (24 pts) Evaluate the following limits.

a) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$. $\left(\frac{0}{0} \text{ type}\right)$

$\overset{\text{L'Hopi Rule}}{=} \lim_{x\to 0} \frac{e^x - 1}{2x}$ $\left(\text{again } \frac{0}{0} \text{ type}\right)$

$\overset{\text{L'Hopis Rule}}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = 1/2$

b) $\lim_{x\to 0^+} (\sin x)^{\frac{2}{\ln x}}$. $= e^{\lim_{x\to 0^+} \frac{2}{\ln x} \ln \sin x} = e^{\lim_{x\to 0^+} \frac{2 \ln \sin x}{\ln x}}$ $\left(\frac{\infty}{\infty} \text{ type}\right)$

$\overset{\text{L'Hopis R.}}{=} e^{\lim_{x\to 0^+} \frac{\frac{2\cos x}{\sin x}}{\frac{1}{x}}} = e^{\lim_{x\to 0^+} \frac{2 \overset{1}{x} \overset{1}{\cos x}}{\sin x}} = e^2$

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c) $\lim_{x\to 0} \frac{\int_0^x \sin(t^2)dt}{x^3}$. $\left(\frac{0}{0} \text{ type}\right)$

$\overset{\text{L'Hopi Rule}}{=} \lim_{x\to 0} \frac{\sin(x^2)}{3x^2} = \frac{1}{3}$

d) $\lim_{x\to\infty} \frac{2^x - 3^x}{2^x + 3^{x+1}}$. $= \lim_{x\to\infty} \frac{\overbrace{\left(\frac{2}{3}\right)^x}^{0} - 1}{\underbrace{\left(\frac{2}{3}\right)^x}_{0} + 3} = \frac{-1}{3}$

Question 4 (16 pts) Find, if they exist, the absolute maximum and the absolute minimum values of $f(x) = \dfrac{x^2+1}{x^2+x+1}$:

a) on the interval $[-2, 2]$.

Since $x^2+x+1 \neq 0\ \forall x$, $f(x)$ is continuous on the closed interval $[-2,2]$ and hence Max/min of f on $[-2,+2]$ exist. To find them we need to check the points in $(-2,+2)$ at which $f' = 0$ and end points $f(-2)$, $f(2)$:

$$f'(x) = \frac{2x(x^2+x+1) - (2x+1)(x^2+1)}{(x^2+x+1)^2} = 0 \iff x^2 - 1 = 0 \iff x = \mp 1 \text{ critical pts.}$$

$f(-2) = \frac{5}{3}$, $\underbrace{f(-1) = 2}_{\text{Abs. MAX.}}$, $\underbrace{f(1) = \frac{2}{3} = \frac{14}{21}}_{\text{Abs. min.}}$, $f(2) = \frac{5}{7} = \frac{15}{21}$

b) on the interval $[1, \infty)$.

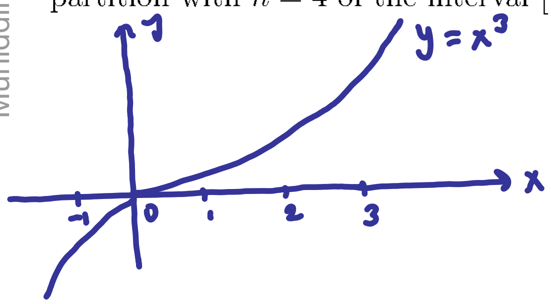
For $x > 1$, $f'(x) > 0$ so f is increasing on $[1, \infty)$

$\lim_{x \to \infty} f(x) = 1$ but $f(x) \neq 1\ \forall x$.

Hence $f(1) = \frac{2}{3}$ is Abs. min on $[1, \infty)$ and there is no abs. MAX.

Question 5 (18 pts) Let $f(x) = x^3$.

a) Write and compute the value of the upper Riemann Sum for f for the equal length partition with $n = 4$ of the interval $[-1, 3]$.



$$U(f,4) = 1\,(f(0) + f(1) + f(2) + f(3))$$
$$= 1 + 8 + 27 = 36$$

b) Write and compute the value of the lower Riemann Sum for f for the equal length partition with $n = 4$ of the interval $[-1, 3]$.

$$L(f,4) = 1\,(f(-1) + f(0) + f(1) + f(2))$$
$$= -1 + 0 + 1 + 8 = 8$$

c) Find $\displaystyle \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^{2/3} + \left(\frac{2}{n}\right)^{2/3} + \cdots + \left(\frac{n-1}{n}\right)^{2/3} + 1\right]$. $= \displaystyle\int_0^1 x^{2/3}\,dx$

$$= \frac{3}{5} x^{5/3} \Big|_0^1 = 3/5$$

Question 6 (20 pts) Let $f(x) = \dfrac{2x^2}{(x+1)^2}$.

a) You are given that $f'(x) = \dfrac{Ax}{(x+1)^3}$ and $f''(x) = \dfrac{B(1-2x)}{(x+1)^4}$.
Show that $20A + 9B + 3 = 119$.

$$f'(x) = \frac{4x(x+1)^{\cancel{2}} - 2\cancel{(x+1)}2x^2}{(x+1)^{\cancel{4}3}} = \frac{4x^2+4x-4x^2}{(x+1)^3} = \frac{4x}{(x+1)^3} \Rightarrow A = 4$$

$$f''(x) = \frac{4(x+1)^{\cancel{3}} - 4x.3\cancel{(x+1)^2}}{(x+1)^{\cancel{6}4}} = \frac{4x+4-12x}{(x+1)^4} = \frac{4-8x}{(x+1)^4} = \frac{4(1-2x)}{(x+1)^4} \Rightarrow B = 4$$

$$A = 4 = B \Rightarrow 20.4 + 9.4 + 3 = 119$$

b) Find the intervals on which f is increasing, f is decreasing, and find the points where f has a local maximum and the points where f has a local minimum.

$f'(x) = 0 \Rightarrow x = 0$ critical point. At $x = -1$, f' is undefined.

x		-1		0	
f'	$+$	‖	$-$	0	$+$
f	↗	‖	↘	0	↗

f is increasing on $(-\infty, -1] \cup [0, +\infty)$
f is decreasing on $[-1, 0]$
At $x = 0$ f has local minimum
f has no local Maximum

c) Find the intervals of concavity and inflection point(s) of f.

$f''(x) = 0 \Rightarrow x = 1/2$ & $f''(-1)$ is undefined

x		-1		$1/2$	
f''	$+$	‖	$+$	0	$-$
f	∪	‖	∪		∩

f is concave up on $(-\infty, -1) \cup (-1, 1/2)$
f is concave down on $(1/2, \infty)$
$x = 1/2$ is inflection point of $f(x)$

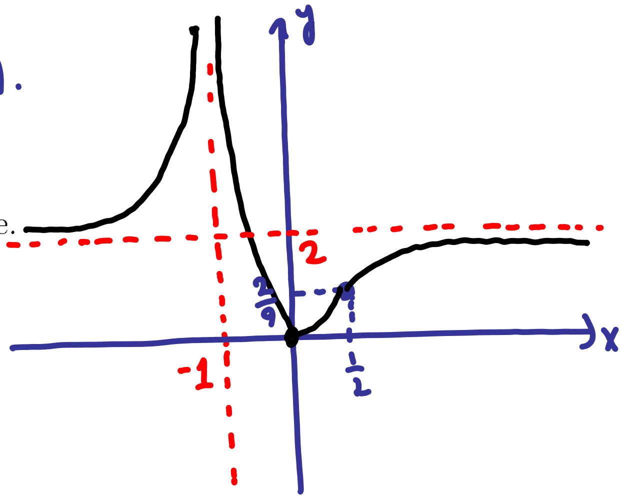
d) Find the horizontal and vertical asymptotes of f.

$\lim_{x \to \mp\infty} f(x) = 2 \Rightarrow y = 2$ is H.A.

$\lim_{x \to -1} f(x) = \infty \Rightarrow x = 1$ is V.A.

e) Sketch the graph of f on the given coordinate plane.

x		-1		0		$1/2$	
f	↗	‖	↘	0	↗		↗
f	∪	‖	∪	0	∪		∩
f	/	‖	\	0	/	$\frac{2}{9}$	(



M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY Final Exam	List No.

Code : *Math 119* Acad. Year : *2010-2011* Semester : *Fall* Coordinator : *Muhiddin Uğuz* Date : *January.13.2011* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK

Question 1 (8 pts) Find an equation of the tangent line to the graph of $f(x) = 1 + \int_0^x \cos(t^2)dt$ at the point $(0, 1)$.

Since $\frac{d}{dx}\int_{s(x)}^{b(x)} g(t)dt = g(b(x)).b'(x) - g(s(x))\,s'(x)$, we have

$\frac{d}{dx}f(x) = f'(x) = 0 + \cos(x^2)\cdot 1 - \cos(0)\cdot 0 = \cos x^2$

$\Rightarrow f'(0) = \cos 0 = 1$

Equation of tangent line to the graph of $y = f(x)$ at the point $(0,1)$ is : $y = f'(0)(x-0) + 1$

$y = 1x + 1$

$y = x + 1$

Question 2 (12 pts) Determine if the following improper integrals are convergent or divergent.

a) $\int_0^{119} \frac{\sin^2 x}{x^{5/2}} dx$

This integral is improper at $x=0$, and so is $\frac{1}{x^{1/2}}$. Since $f(x) = \frac{\sin^2 x}{x^{5/2}} \geq 0$, we can use Limit Comparison (or comparison) Theorem ;

$$\lim_{x\to 0^+} \frac{\frac{\sin^2 x}{x^{5/2}}}{\frac{1}{x^{1/2}}} = \lim_{x\to 0^+} \underbrace{\frac{\sin^2 x}{x^2 \cancel{x^{1/2}}}}_{1} \cdot \frac{\cancel{x^{1/2}}}{1} = 1 \text{ a nonzero finite number}$$

Hence $\int_0^{119} \frac{\sin^2 x}{x^{5/2}} dx$ is convergent if and only if $\int_0^{119} \frac{1}{x^{1/2}} dx$ is convergent.

By p-test, $\int_0^{119} \frac{1}{x^{1/2}} dx$ is conv. $(p = \frac{1}{2} < 1)$

Hence given improper integral is conv.

b) $\int_{120}^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx$

$$= \lim_{c\to\infty} \int_{120}^{c} \frac{1}{x \ln x \ln(\ln x)} dx = \lim_{c\to\infty} \int_{\ln(\ln 120)}^{\ln \ln c} \frac{1}{u} du = \lim_{d\to\infty} \ln u \Big|_{\ln\ln 120}^{d}$$

$$\left(\begin{array}{l} u = \ln(\ln x) \\ du = \frac{1}{\ln x}\frac{1}{x} dx \end{array} \right)$$

$= \infty$

diverges to infinity

Question 3 (20 pts) Evaluate the followings

a) $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2+\sin^2 x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x^2}}{1+\frac{\sin^2 x}{x^2}} = 1$

$\frac{\sin x}{x^2} \to 0$, $\frac{\sin^2 x}{x^2} \to 0$

(Since by squeezing Theorem; $\frac{-1}{x^2} \le \frac{\sin x}{x^2} \le \frac{1}{x^2}$ and $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$)

$\frac{-1}{x^2} \to 0$, $\frac{\sin x}{x^2} \to 0$, $\frac{1}{x^2} \to 0$; $0 \to 0$, $\frac{\sin^2 x}{x^2} \to 0$, $\frac{1}{x^2} \to 0$

b) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x}$ $\left(\frac{0}{0}\text{ type}\right)$

$x-1$ $\ln x$

$\overset{\text{L'Hop's Rule}}{=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1}$ $\left(\frac{0}{0}\text{ type}\right)$

$\overset{\text{L'Hop's Rule}}{=} \lim_{x\to 1} \frac{1}{\ln x + 1 + 1} = 1/2$

$\ln x \to 0$

c) $f'(0)$ where $f(x) = \begin{cases} x^2\sin(\frac{1}{x})+x & \text{if} \quad x\neq 0 \\ 0 & \text{if} \quad x=0 \end{cases}$

$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\(x\neq 0)}} \frac{f(x)}{x} = \lim_{x\to 0} x\sin(\frac{1}{x})+1 = 1$

$-1 \le \sin\frac{1}{x} \le 1$

Hence $-x \le x\sin\frac{1}{x} \le x$ OR $-x \ge x\sin\frac{1}{x} \ge x$

$-x \to 0$, $x\sin\frac{1}{x} \to 0$, $x \to 0$; $-x \to 0$, $x\sin\frac{1}{x} \to 0$, $x \to 0$

since limit exists, we have

$f'(0) = 1$

d) $\frac{dy}{dx}$ implicitly where $x^y + xy = 2011$.

$f(x) = x^y \Rightarrow \ln f = \ln x^y = y.\ln x \Rightarrow$ Take derivative of both sides;

$\frac{f'(x)}{f(x)} = y'\ln x + y\frac{1}{x} \Rightarrow f'(x) = f(x)\left(y'\ln x + \frac{y}{x}\right)$

$\Rightarrow \frac{d}{dx}(x^y) = x^y.\left(y'\ln x + \frac{y}{x}\right)$

Thus

$\left(x^y.y'\ln x + x^y\frac{y}{x}\right) + y + xy' = 0$

$\Rightarrow y' = \frac{-x^y.\frac{y}{x} - y}{x^y\ln x + x}$

Muhiddin UGUZ

Question 4 (20 pts) Evaluate the following:

a) $\displaystyle\int_1^3 \frac{\arctan\sqrt{x}}{\sqrt{x}}dx = 2\int_1^{\sqrt{3}} \arctan w \, dw$

$w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}dx$

$u = \arctan w$, $dv = dw$

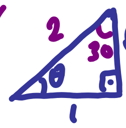
$du = \frac{1}{1+w^2}dw$, $v = w$

$$= 2\left[w \arctan w \Big|_1^{\sqrt{3}} - \underbrace{\int_1^{\sqrt{3}} \frac{w}{1+w^2}dw}\right] = 2\left[\sqrt{3}\cdot\frac{\pi}{3} - \frac{\pi}{4}\right] - \left[\ln 4 - \ln 2\right]$$

$U = 1+w^2$, $du = 2w\,dw$

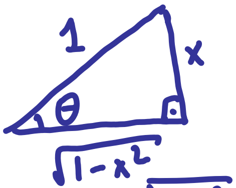
$\frac{1}{2}\int_2^4 \frac{1}{u}du$

$$= \frac{2\sqrt{3}\pi}{3} - \frac{\pi}{2} - \ln 2$$



$\theta = 60° = \pi/3$, $\arctan\sqrt{3} = \frac{\pi}{3}$

b) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}dx$



$\cos\theta = \sqrt{1-x^2}$

$\sin\theta = x$

$\cos\theta \, d\theta = dx$

$$\int \frac{\sin^2\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta = \int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right] + C$$

c) $\displaystyle\int \frac{1}{x^3(x-1)}dx$

$$\frac{1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$(x^2(x-1))$, $(x(x-1))$, $(x-1)$, (x^3)

$\Rightarrow 1 = Ax^3 - Ax^2 + Bx^2 - Bx + Cx - C + Dx^3$

$= x^3(A+D) + x^2(-A+B) + x(-B+C) + (-C)$

$\Rightarrow \boxed{C=-1} \Rightarrow \boxed{B=-1} \Rightarrow \boxed{A=-1} \Rightarrow \boxed{D=1}$

$$\int \frac{1}{x^3(x-1)}dx = \int\left(\frac{-1}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x-1}\right)dx = -\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + \ln|x-1| + C$$

d) $\displaystyle\int \frac{1}{(1+x^{1/3})(1+x^{2/3})}dx = \int \frac{3u^2}{(1+u)(1+u^2)}du$

$\left(u = x^{1/3} \Rightarrow x = u^3, \; dx = 3u^2du\right)$

$$= \int\left(\frac{A}{1+u} + \frac{Bu+C}{1+u^2}\right)du \Rightarrow 3u^2 = Au^2 + A + Bu^2 + Bu + Cu + C$$

$$= u^2\underbrace{(A+B)}_{=3} + u\underbrace{(B+C)}_{=0} + \underbrace{(A+C)}_{=0}$$

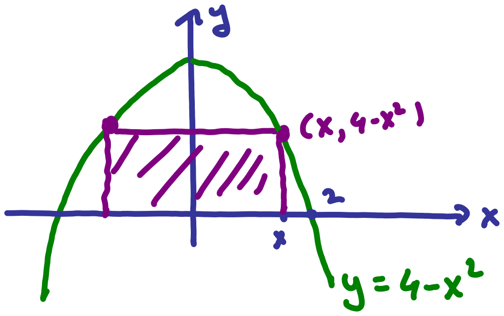
$\Rightarrow \left.\begin{matrix}-C = A\\ -C = B\end{matrix}\right\} A = B \Rightarrow A = B = 3/2$

$C = -3/2$

so, $$\int\left[\frac{3/2}{1+u} + \frac{\frac{3}{2}u - 3/2}{u^2+1}\right]du = \frac{3}{2}\ln|1+u| + \frac{3}{4}\ln|1+u^2| - \frac{3}{2}\arctan u + C$$

$$= \frac{3}{2}\ln|1+x^{1/3}| + \frac{3}{4}\ln|1+x^{2/3}| - \frac{3}{2}\arctan x^{1/3} + C$$

Question 5 (8 pts) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices are above the x-axis on the parabola $y = 4 - x^2$.



Area = Base · height = $2x(4-x^2)$

$A(x) = 8x - 2x^3 \quad x \in [0,2]$

($A(x)$ is a polynomial and hence continuous on the closed interval $[0,2]$. Thus Max, min of $A(x)$ on $[0,2]$ exist.)

To find MAX. of $A(x)$ on $[0,2]$ we must check

1) End points: $A(0) = A(2) = 0$

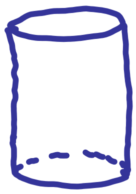
2) singular points in $(0,2)$: That is points at which $A'(x)$ does not exist. Here $A'(x)$ always exits; hence no singular point exists.

3) critical points in $(0,2)$: That is pts at which $A'(x) = 0$

$A'(x) = 8 - 6x^2 = 0 \Rightarrow 6x^2 = 8 \Rightarrow x^2 = \frac{8}{6} = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$ is the only critical point in $(0,2)$; and $A\left(\frac{2}{\sqrt{3}}\right) = 8 \cdot \frac{2}{\sqrt{3}} - 2 \cdot \frac{8}{3\sqrt{3}} = \frac{32}{3\sqrt{3}}$

Thus MAX of $A(x) = \text{MAX}\left\{0, \frac{32}{3\sqrt{3}}\right\} = \frac{32}{3\sqrt{3}} = A(2/\sqrt{3})$ and hence dimensions of Largest-area rectangle are $\frac{4}{\sqrt{3}}$ and $\frac{8}{3}$

Question 6 (8 pts) All dimensions of a right circular cylinder are changing. When the volume is $150cm^3$ it is increasing at the rate of $5cm^3/min$ and at the same moment the radius is $5cm$ and is increasing at the rate of $3cm/min$. Is the height of the cylinder increasing or decreasing? At what rate?



$V(t) = \pi r^2(t) h(t)$

$$5 = \left.\frac{dV}{dt}\right|_{t=t_0} = \pi\left[2\underbrace{r(t_0)}_{=5}\underbrace{r'(t_0)}_{=3}\underbrace{h(t_0)}_{=6/\pi} + \underbrace{r^2(t_0)}_{=25}h'(t_0)\right]$$

when $t = t_0$

$V = 150 = \pi \cdot 5^2 \cdot h(t_0)$

$\Rightarrow h(t_0) = \frac{150}{25\pi} = \frac{6}{\pi}$ cm

$5 = 180 + 25\pi h'(t_0)$

$h'(t_0) = \frac{5 - 180}{25\pi} = \frac{-175}{25\pi} = -\frac{7}{\pi}$ cm/min

Height of the cylinder is decreasing at the rate $\frac{-7}{\pi}$ cm/min

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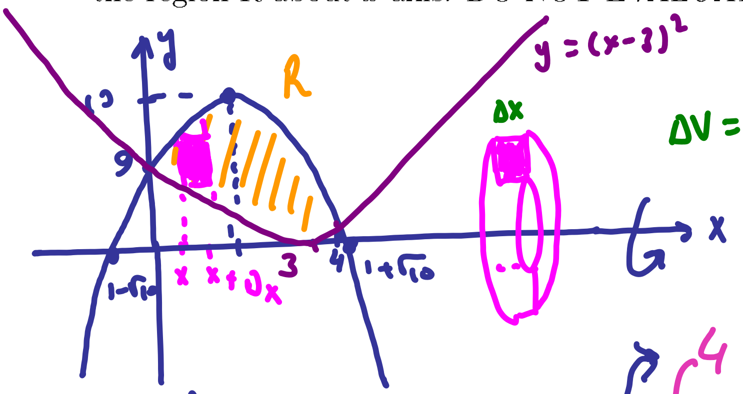
Question 7 (12 pts) Let R be the region bounded by the curves $y = 10 - (x-1)^2$ and $y = (x-3)^2$.

a) Write a definite integral which is equal to the volume of the solid obtained by rotating the region R about x-axis. DO NOT EVALUATE.



$$\Delta V = \Delta x \cdot \pi \cdot (R_{out}^2 - R_{in}^2)$$
$$= \Delta x\, \pi \left((10-(x-1)^2)^2 - (x-3)^4 \right)$$

$y = 10-(x-1)^2$

$y=0 \Rightarrow 10 = (x-1)^2$

$x-1 = \mp\sqrt{10}$

$x = 1 \mp \sqrt{10}$

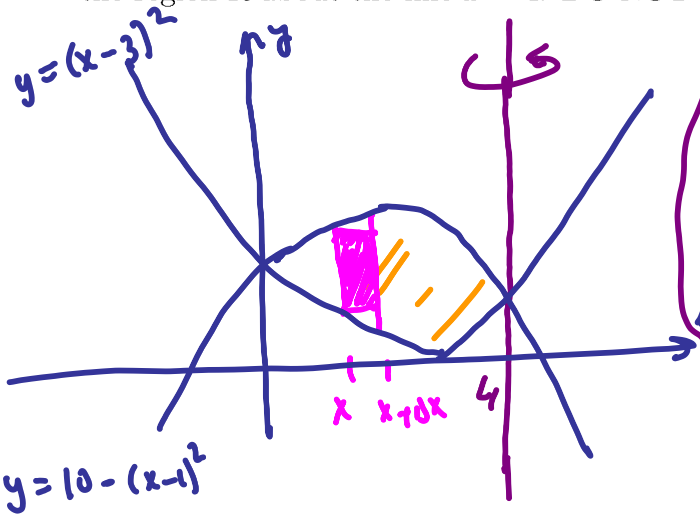
$$V = \int_0^4 \pi \left[\left(10-(x-1)^2\right)^2 - (x-3)^4 \right] dx$$

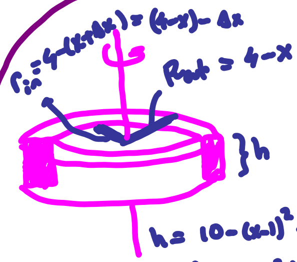
$10-(x-1)^2 = (x-3)$

$10 - x^2 - 1 + 2x = x^2 - 6x + 9$

$0 = 2x^2 - 8x = 2x(x-4)$ $x=0$, $x=4$

b) Write a definite integral which is equal to the volume of the solid obtained by rotating the region R about the line $x = 4$. DO NOT EVALUATE.





$h = 10-(x-1)^2 - (x-3)^2$

$\Delta V = h \cdot \pi (R_{out}^2 - r_{in}^2)$

$= h\pi \left[\cancel{(4-x)^2} - (\cancel{(4-x)^2} + 2(4-x)\Delta x - \underbrace{(\Delta x)^2}_{\approx 0}) \right]$

$= h\pi\, 2(4-x)\Delta x$

OR

cut and open

$2\pi r = 2\pi(4-x)$

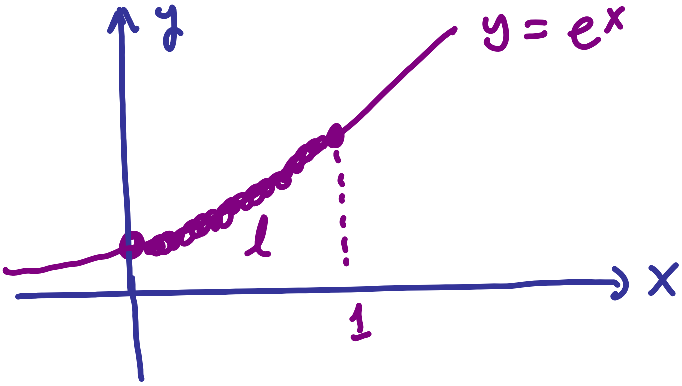
$\Delta V = 2\pi(4-x)\, h\, \Delta x$

$h = 10-(x-1)^2 - (x-3)^2$

$$V = 2\pi \int_0^4 \left[10-(x-1)^2 - (x-3)^2\right](4-x)\, dx$$

Question 8 (12 pts)

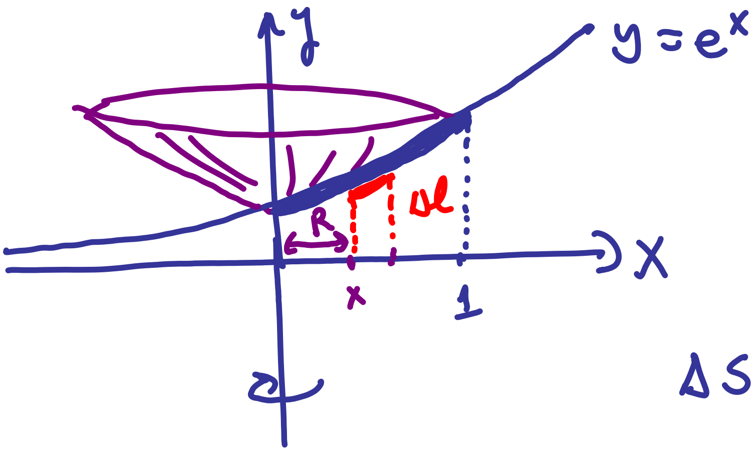
a) Write a definite integral which is equal to the arc length of the curve $y = e^x$ between the points $(0, 1)$ and $(1, e)$. DO NOT EVALUATE.



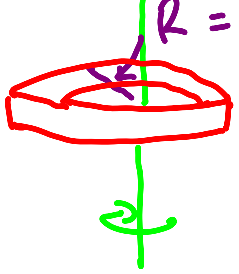
$$\ell = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

b) Write a definite integral which is equal to the area of the surface obtained by rotating about the y-axis the curve $y = e^x$, $0 \le x \le 1$. DO NOT EVALUATE.



$R = x$



$$\Delta\ell = \sqrt{1 + (f'(x))^2}\, \Delta x$$

$$= \sqrt{1 + e^{2x}}\, \Delta x$$

$$\Delta S = 2\pi R\, \Delta\ell$$

$$= 2\pi x \sqrt{1 + e^{2x}}\, \Delta x$$

$$S = 2\pi \int_0^1 x \sqrt{1 + e^{2x}}\, dx$$