Related Rates

Math 117
Spring 2020
METU

(Rate = derivative)

ex: $y = x^3 + 2x$ $\quad \frac{dx}{dt} = 5$. Find $\frac{dy}{dt}$ when $x = 2$.

Solution: Both of x and y are considered as functions of t.
$x = x(t)$ $\quad y = y(t)$. Take derivative of the equation wrt t:

$$\underbrace{\frac{dy}{dt}}_{\text{wanted}} = 3\underbrace{x^2}_{\text{known}}\underbrace{\frac{dx}{dt}}_{\text{known}} + 2\underbrace{\frac{dx}{dt}}_{\text{known}}$$

$$\left.\begin{array}{l} \frac{dx}{dt} = 5 \\ x = 2 \end{array}\right\} \Rightarrow \left.\frac{dy}{dt}\right|_{x=2} = 3 \cdot 2^2 \cdot 5 + 2 \cdot 5 = 70 \quad \blacksquare$$

ex: $x^2 + y^2 = 100$, $\frac{dx}{dt} = 1$. Find $\underbrace{\text{the rate of change in } y}_{=\text{derivative of } y}$, when $x = 6$.
$(x, y \geq 0)$

sol: Differentiate the equation wrt t: $$2\underbrace{x}_{\text{given}}\underbrace{\frac{dx}{dt}}_{\text{known}} + 2\underbrace{y}_{\text{from the equation } x^2+y^2=100}\underbrace{\frac{dy}{dt}}_{\text{wanted}} = 0$$

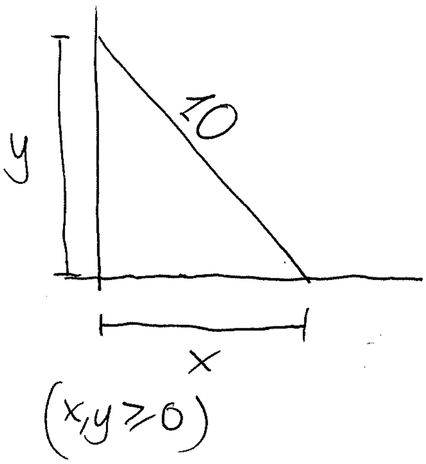
When $x = 6$ y will be $8 \geq 0$.

$$\Rightarrow 2 \cdot 6 \cdot 1 + 2 \cdot 8 \left.\frac{dy}{dt}\right|_{x=6} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-3}{4} \quad \blacksquare$$

In general, the equation is not given. For solution, we must find an equation to differentiate.

ex: A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom is 6 m away from the wall?

solution: A picture helps for writing an equation:



$x^2 + y^2 = 100$

bottom slide rate $= \frac{dx}{dt} = 1$

bottom distance to wall $= x$

top slide rate $= \frac{dy}{dt}$

(negative means slide down)

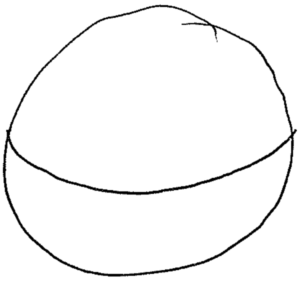
Differentiate wrt t: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

when $x = 6$ $\qquad \left.\frac{dy}{dt}\right|_{x=6} = -\frac{3}{4}$ m/s

Answer is: the top is sliding down 3/4 meters per second.

ex: A snowball is melting so that its surface area decreases at a rate of 1 cm^2/minute. Find the rate at which the diameter (and volume) decreases when the diameter is 10cm.

solution: A snowball is ideally in the shape of a 3 dimensional ball. Its volume is $V=\frac{4}{3}\pi r^3$, its surface area is $A=4\pi r^2$, its diameter is $D=2r$. r, D, A, V are functions of t.



It is given that $\frac{dA}{dt}=-1$ (Decrease means negative)

that is $\frac{d}{dt}(4\pi r^2)=-1$ when $D=10$

$\Rightarrow 4\pi 2r\frac{dr}{dt}=-1$ when $r=5$ $\Rightarrow \frac{dr}{dt}=\frac{-1}{40\pi}$ cm/min when $r=5$.

$\Rightarrow \frac{dD}{dt}=\frac{d}{dt}2r=2\frac{dr}{dt}=\frac{-1}{20\pi}$ when $D=10$.

Diameter decreases at a rate of $1/20\pi$ cm/min when $D=10$.

For Volume: $\frac{dV}{dt}=\frac{4}{3}\pi 3r^2\frac{dr}{dt}$

when $r=5$ $\left.\frac{dV}{dt}\right|_{r=5}=\frac{4}{3}\pi 3\cdot 25\cdot\left.\frac{dr}{dt}\right|_{r=5}=\frac{100\pi}{-40\pi}$ cm^3/min

$=\frac{-5}{3}$ cm^3/min

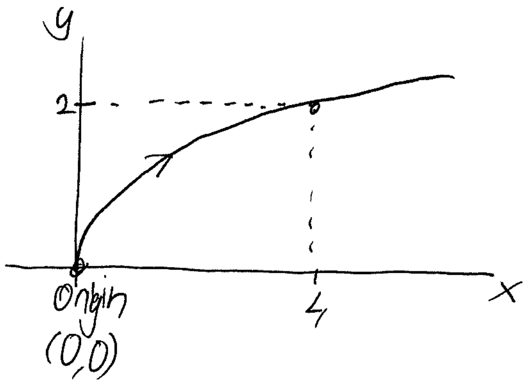
Volume decreases at a rate of $-5/3$ cm^3/min when $D=10$.

ex: A particle is moving along the curve $y=\sqrt{x}$. As the particle passes through the point where $y=2$, its x coordinate increases at a rate of 3 units/second. How fast is the distance from the particle to the origin changing at this instant?

solution: We want to find $\left.\frac{dD}{dt}\right|_{y=2}$, where D is the distance of the particle to the origin.

$$D(x,y)=\sqrt{x^2+y^2}$$

$y=\sqrt{x} \Rightarrow D=\sqrt{x^2+x}$
$(x=y^2)$



Distance $D \geqslant 0$ and there is no ambiguity so we'll work with $D^2=x^2+x$ as it has no squareroot in it.

$\left.\begin{matrix} y=2 \\ y=\sqrt{x} \end{matrix}\right\} \Rightarrow x=4$ when $y=2$.

It is given that $\frac{dx}{dt}=3$ (increase means positive)

Differentiate D^2 wrt t: $\quad 2D\frac{dD}{dt}=2x\frac{dx}{dt}+\frac{dx}{dt}$

When $y=2$ $(x=4)$ D is $\sqrt{20}$ and $2\sqrt{20}\left.\frac{dD}{dt}\right|_{y=2}=2\cdot 4\cdot 3+3$

So $\left.\frac{dD}{dt}\right|_{y=2}$ is $\frac{27}{4\sqrt{5}}$ units/s

4.3 Indeterminate Forms and L'Hospital's Rule

The word "indeterminate" means that the expression is not clear at first sight. Needs work to determine it. Indeterminate types:

$$\underbrace{\frac{0}{0},\ \frac{\infty}{\infty}}_{\text{We use l'Hospital's rule for these.}},\ \underbrace{0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty}_{\text{Others: first converted to the } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ types.}}$$

<u>Theorem</u> (l'Hospital's rule): Assume f, g differentiable and $g'(x) \neq 0$ on an open interval containing a (except possibly at a).

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$,

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit exists.

<u>Note:</u> The theorem can be written as "If $\lim\limits_{x\to a} f(x) = \mp\infty$ and $\lim\limits_{x\to a} g(x) = \mp\infty$

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit exists."

Moreover this rule holds in case of one sided limits and limits at infinity.

<u>ex:</u> $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \underset{\text{by l'Hospital's rule}}{=} \lim\limits_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim\limits_{x\to\infty} 3x^{-1/3} = 0.$

($\frac{\infty}{\infty}$ type)

ex: ($\frac{0}{0}$ type) $\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{\cancel{1/(x+1)}}{1} = 1$

by l'Hospital's rule

ex: $\lim_{x\to\infty} \frac{e^x}{x^2+1} \underset{l'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \underset{l'H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$

($\frac{\infty}{\infty}$ type)

<u>exercise</u>: Show that for any polynomial $p(x)$ (a) $\lim_{x\to\infty} \frac{e^x}{p(x)} = \infty$

(b) $\lim_{x\to\infty} \frac{p(x)}{\ln x} = \infty$

ex: ($0\cdot\infty$ type) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} \underset{l'Hospital's}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$

($\frac{\infty}{\infty}$ type) l'Hospital's ($\frac{\infty}{\infty}$ type)

ex: ($\infty - \infty$ type)

$\lim_{x\to\frac{\pi}{2}-} (\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\frac{\pi}{2}-} \left(\frac{1-\sin x}{\cos x}\right)$

($\frac{0}{0}$ type)

$= \lim_{x\to\frac{\pi}{2}-} \frac{-\cos x}{-\sin x} = 0$

by l'Hospital's

ex: $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0+} \frac{\sin x - x}{x\sin x} \underset{l'H}{=} \lim_{x\to 0+} \frac{\cos x - 1}{\sin x + x\cos x}$

($\infty-\infty$ type) ($\frac{0}{0}$ type) ($\frac{0}{0}$ type)

$\underset{l'H}{=} \lim_{x\to 0+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = 0 .$

ex: $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim\limits_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \underset{\text{by l'Hospital's}}{=} \lim\limits_{x\to\infty} \frac{\frac{-1}{x^2}\cos\left(\frac{1}{x}\right)}{-1/x^2} = 1$

($0\cdot\infty$ type) ($\frac{0}{0}$ type)

(Alternatively, $\lim\limits_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to 0+} \frac{\sin x}{x} = 1$)

ex: $\lim\limits_{x\to 0+} \frac{e^x}{x} = \infty$

We can not use l'Hospital's here because $\frac{1}{0}$ is not one of the indeterminate cases above. If one tries to use it, the result will be wrong.

ex: $\lim\limits_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} = \lim\limits_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) + x^2\frac{-1}{x^2}\cos\left(\frac{1}{x}\right)}{\cos x} = \lim\limits_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) - \overset{??}{\cos\left(\frac{1}{x}\right)}}{\cos x}$

($\frac{0}{0}$ type) l'Hospital's rule does not work.

(The limit is 0:

$\lim\limits_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} = \lim\limits_{x\to 0} \frac{x\sin\left(\frac{1}{x}\right)}{\frac{\sin x}{x}} = \frac{\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = \frac{\overset{\text{by squeeze thm.}}{0}}{1} = 0$)

Indeterminancies of the Form 0^{0}, ∞^{0}, 1^{∞}:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln f(x)^{g(x)}} = \lim_{x\to a} e^{g(x)\ln f(x)}$$

$$\underset{\substack{\text{as } e^x \\ \text{is continuous}}}{=} e^{\lim_{x\to a} g(x)\ln f(x)}$$

ex: $\displaystyle\lim_{x\to 0+}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to 0+} e^{\ln\left(1+\frac{1}{x}\right)^{x}} = \lim_{x\to 0+} e^{x\ln\left(1+\frac{1}{x}\right)}$

(∞^{0} type)

$$\underset{\substack{\text{as } e^x \text{ is} \\ \text{continuous}}}{=} e^{\lim_{x\to 0+} x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to 0+}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}} \underset{\text{by l'Hopital's}}{=} e^{\lim_{x\to 0+}\frac{\frac{-1/x^2}{1+\frac{1}{x}}}{-1/x^2}}$$

($0\cdot\infty$ type) $\quad$ ($\frac{\infty}{\infty}$ type)

$$= e^{\lim_{x\to 0+}\frac{1}{1+\frac{1}{x}}} = e^{0} = 1.$$

exercise Show that $\displaystyle\lim_{x\to 0+}(1+x)^{\frac{1}{x}} = e$.

(1^{∞} type)

ex: $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$ by the exercise above.

exercise: $\displaystyle\lim_{x\to 0+} x^{x} = ?$