Calculus with Analytic Geometry

Recitation 09: Areas of plane regions & solids of revolutions

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Course webpage: [http://ma119.math.metu.edu.tr/](http://ma119.math.metu.edu.tr/)

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Last time: Improper integrals.

Topics to be covered:

- **Ch 7: Applications of Integration**
  - 5.7 Areas of Plane Regions
  - 7.1 Volumes by Slicing-Solids of Revolution

5.7: 3, 5, 9, 11, 15, 17, 19, 21, 23, 29
7.1: 1, 3, 7, 11, 13, 15, 19
7.3: 3, 5, 7, 9, 11, 13, 14, 21, 24, 25, 27, 28, 29
Question 01

Sketch and find the area of the plane region bounded by the curves

1. \( y = (x^2 - 1)^2 \) and \( y = 1 - x^2 \).
2. \( x = y^2 \) and \( x = 2y^2 - y - 2 \).

Note that \( f(x) = \left(1-x^2\right) - \left(x^2-1\right) = 2x^2 - 2x \) is a cone.

Since the graph is symmetric, enough to compute

\[
\text{Area}(A) = \int_0^1 \left(t-x^2\right) - \left(x^2-1\right) \, dx = \int_0^1 \left(-x^4 + x^2\right) \, dx = -\frac{x^5}{5} + \frac{x^3}{3} \bigg|_0^1 = \frac{2}{15}
\]

So, the total area is \( \frac{2}{15} \).

Points of intersection: Points satisfying \( x = y^2 \) and \( x = 2y^2 - y - 2 \).

Above that, \( y^2 = x = 2y^2 - y - 2 \) (\( \iff y^2 - y - 2 = 0 \)) \( \iff y = 2 \) or \( y = -1 \)

Note that \( y^2 = (2y^2 - y - 2) = f(y) \) is not on \( [-1,1] \) (\( x = 4 \) or \( x = 1 \)).

\[
\text{Area}(R) = \int_{-1}^1 f(y) \, dy = \int_{-1}^1 \left[y^2 - (2y^2 - y - 2)\right] \, dy = \int_{-1}^1 -y^2 + y + 2 \, dy = -\frac{y^3}{3} + \frac{y^2}{2} + 2y \bigg|_{-1}^1 = \frac{2}{3}
\]
Question 02

(i) Find the area of the plane region bounded by the curves \( y = \tan x, \ y = 0, \ y = 1 \) and \( x = \frac{\pi}{2} \).

(ii) Find the volume of the solid obtained by rotating the region in part (i) about the \( x \)-axis.

\[
A = \pi \int_{0}^{\frac{\pi}{4}} (\tan x)² \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 \, dx = -\ln(\cos x) \bigg|_{0}^{\frac{\pi}{4}} + \frac{\pi}{4} = \frac{\pi}{4} - \ln \left( \frac{\sqrt{2}}{2} \right).
\]

Recall: How to compute the volume of a solid of revolution:

1. **Disk-method**
   - \( A(x) = \pi \left( R(x)² - r(x)² \right) \)
   - \( V = \int_{a}^{b} A(x) \, dx \)

2. **Shell-method**
   - \( A(x) = 2\pi r(x) h(x) \)
   - \( V = \int_{a}^{b} A(x) \, dx \)

https://www.geogebra.org/m/uGe3Q6YZ#material/qakfj2RB

https://www.geogebra.org/m/uGe3Q6YZ#material/yvbjHFKz
Now, we have

\[ V = \int_{0}^{\pi/4} A_1(x) \, dx + \int_{\pi/4}^{\pi/2} \, A_2(x) \, dx \]

\[ = \pi \int_{0}^{\pi/4} \cos^2 x \, dx + \pi \int_{\pi/4}^{\pi/2} \sqrt{1 - \sin^2 x} \, dx \]

\[ = \pi. \]

(b) Using Shell-method:

\[ V = \int_{0}^{1} A_1(y) \, dy + \int_{1}^{\infty} A_2(y) \, dy \]

\[ = \int_{0}^{1} 2\pi y \left( \sqrt{\frac{\pi}{2} - \arctan y} \right) \, dy + \int_{1}^{\infty} 2\pi y \left( \frac{\pi}{2} - \arctan y \right) \, dy \]

\[ = \ldots. \]
Find the volume of the solid if the plane region bounded by \( y = 1 + \sin x \) and \( y = 1 \) from \( x = 0 \) to \( x = \pi \) rotated about

(i) the \( x \)-axis,
(ii) the \( y \)-axis.

\[ A = \pi [R^2(x) - r^2(x)] \]

(i) Here, \( R(x) = \sin x + 1 \)
\[ r(x) = 1 \]

Hence we have
\[ V = \int_{0}^{\pi} A(x) \, dx \]
\[ = \int_{0}^{\pi} \pi \left( (\sin x + 1)^2 - 1^2 \right) \, dx \]
\[ = \pi \int_{0}^{\pi} (\sin^2 x + 2\sin x) \, dx \]
\[ = \pi \left[ \frac{\pi}{2} + 2 \right] \]

(ii) Here, \( h(x) = \sin x + 1 - 1 = \sin x \)
\[ r(x) = x \]

and \( A(x) = 2\pi r(x)h(x) = 2\pi x\sin x \).

Then we get
\[ V = \int_{0}^{\pi} A(x) \, dx = \int_{0}^{\pi} 2\pi x \sin x \, dx \]
[we use IBP with \( u = x \), \( dv = \sin x \) dx]
\[ = 2\pi \left[ -x \cos x \right]_{0}^{\pi} \]
\[ = 2\pi \left[ \sin x - x \cos x \right]_{0}^{\pi} \]
\[ = 2\pi \left[ 0 - 0 \right] \]
\[ = 2\pi \]

Exercise: Using Shell-method.
The triangular region with vertices $(0, -1), (1, 0)$ and $(0, 1)$ is rotated about the line $x = 2$. Find the volume of the solid generated.

\[ \text{Solu} \]

\[ \text{exercize: Use disk method} \]

\[ \text{L, start with horizontal slice} \]

\[ \text{here,} \quad k(x) = 1-x - (x-1) = 2(1-x) \]

\[ r(x) = 2-x \]

\[ \text{and hence the cross-sectional area becomes} \]

\[ A(x) = 2\pi r(x) h(x) = 2\pi (2-x)(2-2x) \]

\[ \text{Then the volume is given by} \]

\[ V = \int A(x) \, dx = \int (2-x)(1-x) \, dx = \frac{10\pi}{3} \]
Find the volume of the solid obtained by rotating the region bounded by \( y = 4 - x^2 \) and \( y = 0 \) about the line

(i) \( x = 3 \),
(ii) \( y = 6 \).

\[
\text{Volume} = 2\pi \int_{-2}^{2} (3 - x) (4 - x^2) \, dx
\]

The cross-sectional area is:

\[
A(x) = \pi r^2(x) - \pi r(x) = \pi \left( \frac{36}{5} - (x^2) \right)
\]

The volume of the solid is:

\[
V = \int_{-2}^{2} A(x) \, dx = \pi \left( \frac{36x - x^4}{5} - \frac{2x^3 - 4x}{3} \right) \bigg|_{-2}^{2}
\]