MATHEMATICS DEPARTMENT
MIDDLE EAST TECHNICAL UNIVERSITY
ANKARA — TURKEY

MATH 117 - CALCULUS I

WEEK 7-8: Chapter 3: Transcendental Functions

3.1 Inverse Functions
3.2 Exponential and Logarithmic Functions
3.3 The Natural Logarithm and Exponential
3.5 The Inverse Trigonometric Functions

3.1 Inverse Functions

Definition: A function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$, or equivalently if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Definition: If f is one-to-one, then it has an inverse function f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$.

$$y = f^{-1}(x) \Leftrightarrow x = f(y)$$

Properties

1. The domain of f^{-1} is the range of f, the range of f^{-1} is the domain of f.
2. $f^{-1}(f(x)) = x$, $f(f^{-1}(x)) = x$, $(f^{-1})^{-1}(x) = f(x)$
3. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.

Derivatives

The derivative of $y = f^{-1}(x)$:

$y = f^{-1}(x) \Leftrightarrow x = f(y)$ differentiate both sides with respect to x, assuming y is a function of x:

$$1 = f'(y) \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} \quad \text{and} \quad y = f^{-1}(x)$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

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1. Show that the following functions are one-to-one, find the inverses of the functions and specify the domains and ranges of f and f^{-1}.

(a) $f(x) = \sqrt{2x+1}$

Solution:

(i) Domain and Range of f:
f is defined when $2x+1 \geq 0 \Rightarrow x \geq -\frac{1}{2}$, so the domain of f is $D(f) = [-\frac{1}{2}, \infty)$
Range of f is $R(f) = [0, \infty)$

(ii) Is f one-to-one?
Let $x_1, x_2 \in D(f) = [-\frac{1}{2}, \infty)$ and assume that $f(x_1) = f(x_2)$

$\sqrt{2x_1+1} = \sqrt{2x_2+1} \Rightarrow 2x_1+1 = 2x_2+1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Thus, f is one-to-one

(iii) Since f is one-to-one, it has the inverse $f^{-1} : [0, \infty) \longrightarrow [-\frac{1}{2}, \infty)$

$D(f^{-1}) = [0, \infty)$ and $R(f^{-1}) = [-\frac{1}{2}, \infty)$.

To find the inverse of f, solve $y = \sqrt{2x+1}$ for x

$y^2 = 2x+1 \Rightarrow x = \dfrac{y^2-1}{2}$

So, $f^{-1}(x) = \dfrac{x^2-1}{2}$ is the inverse of f.

(b) $f(x) = x^2 - 4$, $x > 0$

Solution:

(i) Domain and Range of f
The domain is given as $D(f) = (0, \infty)$ and Range is $R(f) = (-4, \infty)$

(ii) Is f one-to-one?
Observe that $f'(x) = 2x > 0$ on the domain of f $(x > 0)$, so f is strictly increasing, it must be one-to-one.

(iii) Since f is one-to-one, it has the inverse $f^{-1} : (-4, \infty) \rightarrow (0, \infty)$

$y = x^2 - 4 \Rightarrow x^2 = y + 4 \Rightarrow x = \sqrt{y+4}$

$f^{-1}(x) = \sqrt{x+4}$ is the inverse of f

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(c) $f(x) = \dfrac{1-2x}{1+x}$

Solution:

(i) Domain of f :

$f(x)$ is defined when $1+x \neq 0$, $x \neq -1$, $D(f) = \mathbb{R} - \{-1\}$

(ii) Is f one-to-one ?

Let $x_1, x_2 \in D(f)$ and assume $f(x_1) = f(x_2)$.

$$\frac{1-2x_1}{1+x_1} = \frac{1-2x_2}{1+x_2} \Rightarrow \cancel{1} + x_2 - 2x_1 - \cancel{2x_1 . x_2} = \cancel{1} + x_1 - 2x_2 - \cancel{2x_1 . x_2}$$

$\Rightarrow 3x_2 = 3x_1 \Rightarrow x_1 = x_2$

f is one-to-one.

(iii) Since f is one-to-one , it has the inverse $f^{-1} : R(f) \rightarrow D(f)$

$$y = \frac{1-2x}{1+x} \Rightarrow y + xy = 1-2x \Rightarrow 2x + xy = 1-y \Rightarrow x(2+y) = 1-y \Rightarrow x = \frac{1-y}{2+y}$$

$f^{-1}(x) = \dfrac{1-x}{2+x}$ and domain of f^{-1} is $\mathbb{R} - \{-2\}$

Thus ; $D(f) = R(f^{-1}) = \mathbb{R} - \{-1\}$ and $D(f^{-1}) = R(f) = \mathbb{R} - \{-2\}$

2. Find $(f^{-1})'(-2)$ where $f(x) = x^3 + x - 4$.

Solution:

The derivative of the inverse function is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$f'(x) = 3x^2 + 1$

$f^{-1}(-2) = x \Rightarrow f(x) = -2 \quad x = 1 \quad , f(1) = 1+1-4 = -2$

Thus, $(f^{-1})'(-2) = \dfrac{1}{f'(f^{-1}(-2))} = \dfrac{1}{f'(1)} = \dfrac{1}{3\,1^2+1} = \dfrac{1}{4}$.

3. Show that the function $g(x) = \sin x + 2x$ is one-to-one and find $(g^{-1})'(2\pi)$.

<u>Solution:</u>

The derivative of g is $g'(x) = \cos x + 2$ and observe that for all x $-1 \leqslant \cos x \leqslant 1$.
To get $g'(x)$: $-1+2 \leqslant \cos x + 2 \leqslant 1+2 \Rightarrow 1 \leqslant g'(x) \leqslant 3$
Thus, for all x, $g'(x) > 0$ so $g(x)$ is a strictly increasing function, it must be one-to-one.

$$(g^{-1})'(2\pi) = \frac{1}{g'(g^{-1}(2\pi))}$$

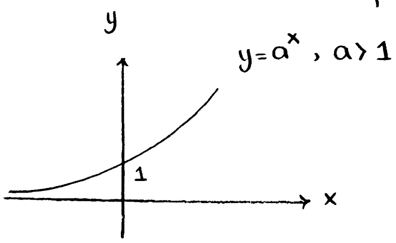
$g'(x) = \cos x + 2$

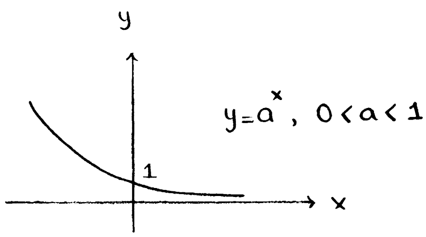
$g^{-1}(2\pi) = x \Rightarrow g(x) = 2\pi$ when $x = \pi$, $g(\pi) = \sin \pi + 2\pi = 2\pi$ so $g^{-1}(2\pi) = \pi$

Thus; $$(g^{-1})'(2\pi) = \frac{1}{g'(g^{-1}(2\pi))} = \frac{1}{g'(\pi)} = \frac{1}{\cos \pi + 2} = 1$$

3.2 Exponential and Logarithmic Functions

Exponential Functions: An exponential function is a function of the form $f : \mathbb{R} \to \mathbb{R}^+$ $f(x) = a^x$ where a is a positive constant.





Laws of Exponents: Let $a > 0$, $b > 0$, $x, y \in \mathbb{R}$

(i) $a^0 = 1$

(ii) $a^{x+y} = a^x \cdot a^y$

(iii) $a^{-x} = \dfrac{1}{a^x}$

(iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$

(vi) $(ab)^x = a^x \cdot b^x$

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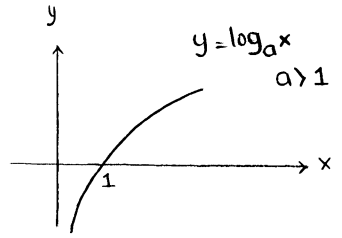
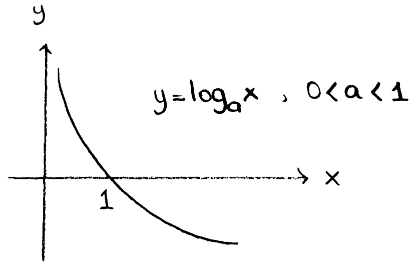
Logarithms: If $a>0$ and $a \neq 1$, the function $\log_a x$ is the inverse of a^x:

$y = \log_a x \Leftrightarrow x = a^y$

The logarithmic function is defined as $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_a x$, $a>0$, $a \neq 1$.





Laws of Logarithms: let $x>0$, $y>0$, $a>0$, $b>0$ and $a \neq 1$, $b \neq 1$

(i) $\log_a 1 = 0$

(ii) $\log_a (xy) = \log_a x + \log_a y$

(iii) $\log_a (x^y) = y . \log_a x$

(iv) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a x = \dfrac{\log_b x}{\log_b a}$

3.3 The Natural Logarithm and Exponential

$e = 2.71828.....$

The Natural Logarithm: $f: \mathbb{R}^+ \to \mathbb{R}$ $f(x) = \log_e x = \ln x$

The Exponential Function: It is the inverse of the Natural Logarithm, $f: \mathbb{R} \to \mathbb{R}^+$, $f(x) = e^x$.

$y = \ln x \Leftrightarrow x = e^y$

Derivatives

(i) $\dfrac{d}{dx} a^x = a^x . \ln a$

(ii) $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \ln a}$

(iii) $\dfrac{d}{dx} e^x = e^x$

(iv) $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$

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4. Solve the following equations.

(a) $\log_5(10+x) = \log_5(3+4x)$.

Solution:

Since the logarithmic function is one-to-one $10+x = 3+4x \Rightarrow 3x = 7 \Rightarrow x = \frac{7}{3}$.

(b) $\log_2(x+1) - \log_2(x-1) = 1$

Solution:

$\log_2(x+1) - \log_2(x-1) = 1 \Rightarrow \log_2\left(\frac{x+1}{x-1}\right) = 1 \Rightarrow \frac{x+1}{x-1} = 2^1 \Rightarrow x+1 = 2x-2 \Rightarrow x = 3$.

(c) $3\log_5 x + \log_{25} x = 14$.

Solution:

$\log_5 x^3 + \frac{1}{2}\log_5 x = 14 \Rightarrow \log_5 x^3 + \log_5 x^{1/2} = 14 \Rightarrow \log_5 x^{7/2} = 14 \Rightarrow x^{7/2} = 5^{14} \Rightarrow x = 5^{14 \cdot \frac{2}{7}} = 5^4$

5. Find the derivatives of the following functions.

(a) $f(x) = e^{\sin(x^3-3)}$.

Solution:

$f'(x) = e^{\sin(x^3-3)} \cdot \cos(x^3-3) \cdot 3x^2$

(b) $f(x) = \dfrac{\ln(\tan x)}{e^{\tan x}}$

Solution:

$$f'(x) = \frac{\frac{1}{\tan x}(1+\tan^2 x) \cdot e^{\tan x} - e^{\tan x} \cdot (1+\tan^2 x) \cdot \ln(\tan x)}{e^{2\tan x}}$$

(c) $f(x) = \ln(3^{x^2+1} \cdot \ln x)$

Solution:

$$f'(x) = \frac{1}{3^{x^2+1} \cdot \ln x} \cdot \left(3^{x^2+1} \cdot \ln 3 \cdot (2x) \cdot \ln x + 3^{x^2+1} \cdot \frac{1}{x}\right)$$

(d) $f(x) = \log_2(e^{x^2} + x)$

Solution:

$$f'(x) = \frac{1}{(e^{x^2}+x)\ln 2} \cdot \left(e^{x^2} \cdot (2x) + 1\right)$$

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Logarithmic Differentiation

Suppose we want to differentiate a function of the form $y = [f(x)]^{g(x)}$ for $f(x) > 0$.
The derivative can be obtained by taking natural logarithm of both sides:

$y = [f(x)]^{g(x)}$, $f(x) > 0 \Rightarrow \ln y = \ln [f(x)]^{g(x)}$

$\ln y = g(x) . \ln [f(x)]$ differentiate with respect to x, assuming y is a function of x

$$\frac{y'}{y} = g'(x) . \ln [f(x)] + g(x) . \frac{f'(x)}{f(x)}$$

Thus, the derivative of $y = [f(x)]^{g(x)}$ is

$$y' = [f(x)]^{g(x)} . \left(g'(x) . \ln [f(x)] + g(x) . \frac{f'(x)}{f(x)} \right)$$

6. Find the derivatives of the following functions:

(a) $f(x) = \left(\frac{1}{x}\right)^{\ln x}$

Solution:

For the values x such that $f(x) > 0$, take the natural logarithm of both sides:

$\ln f(x) = \ln \left[\left(\frac{1}{x}\right)^{\ln x} \right] \Rightarrow \ln f(x) = \ln x . \ln\left(\frac{1}{x}\right)$ differentiate with respect to x

$$\frac{f'(x)}{f(x)} = \frac{1}{x} . \ln\left(\frac{1}{x}\right) + \ln x . \ x \left(-\frac{1}{x^2}\right) \Rightarrow f'(x) = \left(\frac{1}{x}\right)^{\ln x} \left[\frac{1}{x} . \ln\left(\frac{1}{x}\right) + \ln x \left(-\frac{1}{x}\right) \right]$$

(b) $f(x) = (x^2+1)^{\sqrt[3]{x}}$

Solution:

Take the natural logarithm of both sides; $\ln f(x) = \ln \left[(x^2+1)^{\sqrt[3]{x}} \right]$

$\ln f(x) = x^{1/3} . \ln (x^2+1)$ differentiate with respect to x:

$$\frac{f'(x)}{f(x)} = \frac{1}{3} \ x^{-2/3} . \ln (x^2+1) + x^{1/3} . \frac{2x}{x^2+1}$$

$$f'(x) = (x^2+1)^{x^{1/3}} \left[\frac{1}{3} \ x^{-2/3} . \ln (x^2+1) + \frac{2 \ x^{4/3}}{x^2+1} \right]$$

(c) $f(x) = x^{\sqrt{x}}$

Solution:

For $x > 0$, take the natural logarithm of both sides: $\ln f(x) = \ln [x^{\sqrt{x}}]$

$\ln f(x) = \sqrt{x} . \ln x$ differentiate with respect to x

$$\frac{f'(x)}{f(x)} = \frac{1}{2\sqrt{x}} . \ln x + \sqrt{x} . \frac{1}{x} \Rightarrow f'(x) = x^{\sqrt{x}} \left[\frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} \right]$$

7- Use logarithmic differentiation to find the derivative of the function $f(x) = \dfrac{\sin^2 x \cdot \tan^4 x}{(x^2+1)^2}$.

Solution:

For $f(x)>0$; take natural logarithm of $f(x)$:

$$\ln f(x) = \ln\left[\frac{\sin^2 x \cdot \tan^4 x}{(x^2+1)^2}\right] \Rightarrow \ln f(x) = \ln(\sin^2 x) + \ln(\tan^4 x) - \ln((x^2+1)^2)$$

$\ln f(x) = 2 . \ln(\sin x) + 4 \ln(\tan x) - 2\ln(x^2+1)$ differentiate both sides with respect to x

$$\frac{f'(x)}{f(x)} = 2\frac{\cos x}{\sin x} + 4\frac{1+\tan^2 x}{\tan x} - 2\frac{2x}{(x^2+1)}$$

$$f'(x) = \frac{\sin^2 x . \tan^4 x}{(x^2+1)^2}\left[2\cot x + 4\left(\frac{1+\tan^2 x}{\tan x}\right) - \frac{4x}{x^2+1}\right]$$

8. Let $f(x) = \dfrac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}$, find $f'(2)$ and $f'(1)$.

Solution:

To find $f'(2)$, use logarithmic differentiation, for $f(x)>0$

$$\ln f(x) = \ln\left[\frac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}\right]$$

$$\ln f(x) = \ln(x^2-1) + \ln(x^2-2) + \ln(x^2-3) - \ln(x^2+1) - \ln(x^2+2) - \ln(x^2+3)$$

$$\frac{f'(x)}{f(x)} = \frac{2x}{x^2-1} + \frac{2x}{x^2-2} + \frac{2x}{x^2-3} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2} - \frac{2x}{x^2+3}$$

let $x=2$; $f'(2) = f(2) . 4\left[\frac{1}{3} + \frac{1}{2} + 1 - \frac{1}{5} - \frac{1}{6} - \frac{1}{7}\right] = \dfrac{556}{3675}$

$$f(2) = \frac{3.2.1}{5.6.7}$$

To find $f'(1)$; $f(x) = (x^2-1) . \underbrace{\frac{(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}}_{g(x)} = (x^2-1) . g(x)$

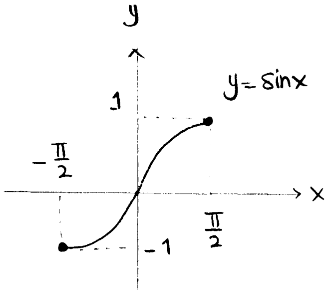
$$f'(x) = 2x . g(x) + (x^2-1) . g'(x)$$

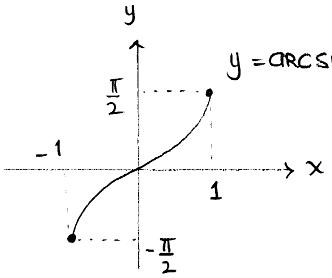
$$f'(1) = 2 . g(1) + \underbrace{0 . g'(1)}_{0} = 2 . \frac{(-1)(-2)}{2.3.4} = \frac{1}{6}$$

3.5 The Inverse Trigonometric Functions

To define inverse trigonometric functions, we consider trigonometric function on some restricted domains on which they are one-to-one.

The Arcsine Function: The restricted function $\sin x$, $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$, $f(x) = \sin x$ is one-to-one, so it has the inverse $f^{-1}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f^{-1}(x) = \arcsin x$.





$y = \arcsin x \Leftrightarrow$
$x = \sin y$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

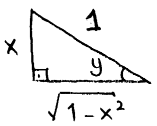
The derivative of Arcsin:

$y = \arcsin x \Leftrightarrow x = \sin y$ differentiate with respect to x

$1 = \cos y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

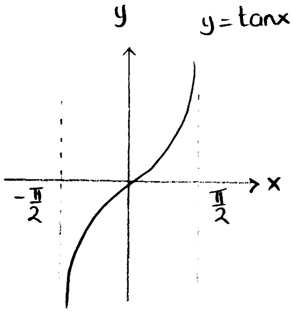
$$\boxed{\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

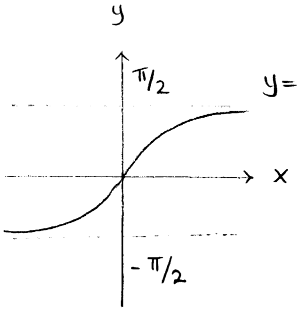
$x = \sin y \Rightarrow$



$\cos y = \frac{\sqrt{1-x^2}}{1}$

The Arctangent Function: The restricted function $\tan x$, $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x) = \tan x$ is one-to-one, so it has the inverse $f^{-1}: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f^{-1}(x) = \arctan x$.





$y = \arctan x \Leftrightarrow$
$x = \tan y$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

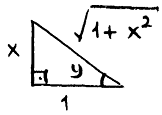
The derivative of Arctan:

$y = \arctan x \Leftrightarrow x = \tan y$ differentiate with respect to x

$1 = \sec^2 y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx} \arctan x = \frac{1}{1+x^2}}$$

$x = \tan y \Rightarrow$



$\sec y = \sqrt{1+x^2}$

The Arccosine Function: $f : [-1,1] \to [0,\pi]$ $f(x) = \arccos x$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

9. Evaluate the given expressions:

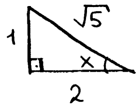
(a) $\cos\left(\arctan \frac{1}{2}\right)$.

Solution:

Let $x = \arctan \frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

We are looking for $\cos x$:

$\tan x = \frac{1}{2}$



$\cos x = \frac{2}{\sqrt{5}}$

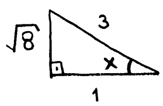
(b) $\sin\left(\arccos\left(-\frac{1}{3}\right)\right)$

Solution:

Let $x = \arccos -\frac{1}{3} \Rightarrow \cos x = -\frac{1}{3}$ $x \in [0, \pi]$ since the value is negative $x \in \left[\frac{\pi}{2}, \pi\right]$

We are looking for $\sin x$:

$\cos x = \left|-\frac{1}{3}\right|$



$\sin x = \frac{\sqrt{8}}{3}$ and on $\left[\frac{\pi}{2}, \pi\right]$ it is positive.

10. Find the derivatives of the following functions

(a) $f(x) = \arctan(x^2+1)$

Solution:

$$f'(x) = \frac{1}{1+(x^2+1)^2} \cdot 2x$$

(b) $f(x) = x - \arcsin(\sqrt{x}+1)$

Solution:

$$f'(x) = 1 - \frac{1}{\sqrt{1-(\sqrt{x}+1)^2}} \cdot \frac{1}{2\sqrt{x}}$$

(c) $f(x) = e^{\arcsin(x^2)}$

Solution:

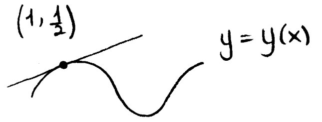
$$f'(x) = e^{\arcsin(x^2)} \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x$$

11 - Find an equation for the tangent line to the graph of the function $y = y(x)$ which is given implicitly by $\arcsin(xy) = x^2 y + 5x - \frac{11}{2} + \frac{\pi}{6}$ at the point $(1, \frac{1}{2})$.

Solution:

Check that $(1, \frac{1}{2})$ is on the curve $y = y(x)$.

$\arcsin(1 \cdot \frac{1}{2}) \stackrel{?}{=} 1 \cdot \frac{1}{2} + 5 - \frac{11}{2} + \frac{\pi}{6} \Rightarrow \frac{\pi}{6} = \frac{\pi}{6}$ ✓ This point is on the curve. $y(1) = \frac{1}{2}$



The slope of the tangent line is $y'(1)$.

Differentiate $\arcsin(xy) = x^2 y + 5x - \frac{11}{2} + \frac{\pi}{6}$ implicitly.

$$\frac{1}{\sqrt{1 - x^2 y^2}} \cdot (y + x y') = 2xy + x^2 y' + 5$$

Let $x = 1$ and $y(1) = \frac{1}{2}$

$$\frac{1}{\sqrt{1 - (y(1))^2}} \left(y(1) + y'(1) \right) = 2 \cdot 1 \cdot y(1) + y'(1) + 5 \Rightarrow y'(1) = \frac{6\sqrt{3} - 1}{2 - \sqrt{3}}$$

The equation of the tangent line $y - \frac{1}{2} = \frac{6\sqrt{3} - 1}{2 - \sqrt{3}} (x - 1)$.