MATH 117 & MATH 119

RECITATION 4

Chapter 4 : More Applications of Differentiation

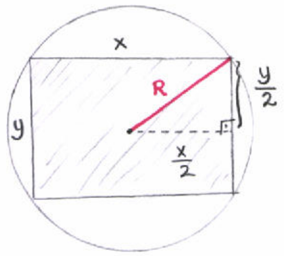
4.8 Extreme - Value Problems

4.9 Linear Approximations

4.8 Extreme - Value Problems

1. Find the dimensions of a rectangle of largest area that can be inscribed in a circle of radius R.

Solution:



The area of the rectangle is $A = x.y$

Firstly, by using the relation between x, y and R, we need to write the area as a function of x and maximize $A(x)$.

$R^2 = \frac{x^2}{4} + \frac{y^2}{4} \Rightarrow y = \sqrt{4R^2 - x^2}$, $x \in [0, 2R]$ (If $x=0$ or $x=2R$ the area can be considered as 0)

The area of the rectangle is $A(x) = x.\sqrt{4R^2 - x^2}$, $x \in [0, 2R]$

For simplicity, find the absolute maximum value of $[A(x)]^2 = f(x)$

$f(x) = x^2.(4R^2 - x^2)$ is continuous on $[0, 2R]$ so it has absolute maximum and minimum values.

$f'(x) = 2x(4R^2 - x^2) + x^2(-2x) = 8R^2x - 4x^3 = 4x(2R^2 - x^2) = 4x(\sqrt{2}R - x)(\sqrt{2}R + x)$

$f'(x) = 0 \Rightarrow$ $x = 0 \in D(f)$, $x = \sqrt{2}.R \in D(f)$ } are critical points; $x = -\sqrt{2}.R \notin D(f)$

$x=0$, $x=2R$ are the endpoints of the domain.

x	0		$\sqrt{2}R$		2R
f'(x)		+		−	
f(x)		↗	maximum (absolute)	↘	

When $x = \sqrt{2}.R$, $[A(x)]^2$ has its absolute maximum value, so $A(x)$ has also absolute maximum value at $x = \sqrt{2}.R$

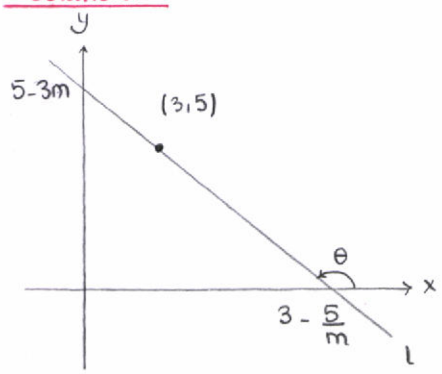
The dimensions of the rectangle : $x = \sqrt{2}.R$

$y = \sqrt{4R^2 - x^2} = \sqrt{4R^2 - 2R^2} = \sqrt{2}R$

($x = y$, when it is a square, it has maximum area)

2. Find the equation of the line passing through (3,5) cuts of the least area from the first quadrant.

Solution:



To write the equation of the line, we need to find the slope of this line

$\tan\theta = m$ slope

Then the equation can be written as

$y - 5 = m(x-3)$

Intercepts of the line:

$x = 0 \Rightarrow y = 5 - 3m$

$y = 0 \Rightarrow x = 3 - \frac{5}{m}$

Observe that $\frac{\pi}{2} < \theta < \pi \Rightarrow \tan\theta < 0 \Rightarrow m \in (-\infty, 0)$

Then the area of the region can be written as a function of m:

$f(m) = \frac{1}{2} \cdot (5-3m)\left(3 - \frac{5}{m}\right) = \frac{1}{2}\left(30 - 9m - \frac{25}{m}\right) \qquad m \in (-\infty, 0)$

f is continuous on $(-\infty, 0)$, $\lim\limits_{m \to -\infty} f(m) = \infty$, $\lim\limits_{m \to 0^-} f(m) = \infty$, f has no absolute maximum value, but it has absolute minimum value.

$f'(m) = \frac{1}{2}\left(-9 + \frac{25}{m^2}\right) = \frac{1}{2}\left(\frac{5}{m} + 3\right)\left(\frac{5}{m} - 3\right)$

$f'(m) = 0 \Rightarrow \quad m = -\frac{5}{3} \in D(f)$

$\qquad\qquad\quad m = \frac{5}{3} \notin D(f)$

m		$-\frac{5}{3}$		0
$f'(m)$	$-$		$+$	
	$\searrow$	minimum	$\nearrow$	

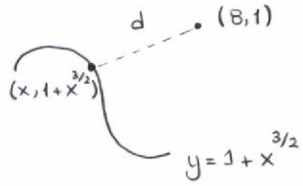
If $m = -\frac{5}{3}$, f attains its absolute minimum value.

The equation of the line passing through (3,5) with slope $m = -\frac{5}{3}$:

$y - 5 = -\frac{5}{3}(x-3)$.

3. Find the shortest distance from the point $(8,1)$ to the curve $y = 1 + x^{3/2}$.

Solution:



The curve is defined for $x \geq 0$.

The distance between $(8,1)$ and an arbitrary point on the curve $(x, 1+x^{3/2})$ is

$d(x) = \sqrt{(x-8)^2 + (1+x^{3/2}-1)^2}$, $x \geq 0$

We want to minimize $d(x)$, for simplicity find the absolute minimum of $f(x) = [d(x)]^2$ on $[0, \infty)$

$f(x) = (x-8)^2 + x^3 = x^3 + x^2 - 16x + 64$

($\lim_{x \to \infty} f(x) = \infty$, f has no absolute maximum value, but it has absolute minimum value)

$f'(x) = 3x^2 + 2x - 16 = (3x+8)(x-2)$

$f'(x) = 0 \Rightarrow x = -8/3 \notin D(f)$
$\qquad\qquad\quad x = 2 \in D(f)$

x		0		2	
f'			$-$		$+$
			$\searrow$	$\smile$	$\nearrow$

local minimum

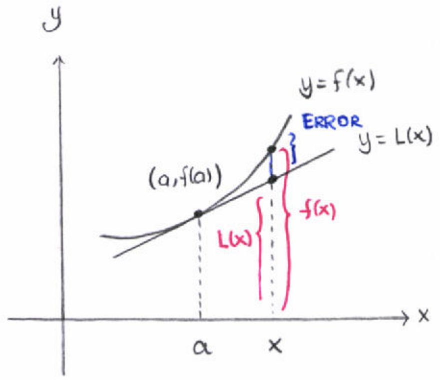
(Compare endpoint and critical point)

$f(2) = 8 + 4 - 32 + 64 = 44 \Rightarrow d(2) = \sqrt{44}$ $\rightarrow$ absolute minimum

$f(0) = 64 \Rightarrow d(2) = \sqrt{64} = 8$

The shortest distance from $(8,1)$ to $y = 1 + x^{3/2}$ is $\sqrt{44}$.

4.9 Linear Approximations



The linearization of the function f about a is the function L defined by

$L(x) = f(a) + f'(a)(x-a)$

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$ provides linear approximations for values of f near a.

4. Let f be a differentiable function such that $f(1)=0$ and $x^2+xf(x)+[f(x)]^2=1$. Estimate $f(1.01)$ using linear approximation.

Solution:

The point 1.01 is close to 1, so the linearization of f about $x=1$ provides a linear apprizimation for $f(1.01)$.

The linearization of f about $x=1$; $L(x)=f(1)+f'(1)\,(x-1)$

To find $f'(1)$, differentiate $x^2+x.f(x)+[f(x)]^2=1$ implicitly

$2x+f(x)+xf'(x)+2f(x).f'(x)=0$, put $x=1$, $f(1)=0$

$2+\underbrace{f(1)}_{0}+f'(1)+2\underbrace{f(1)}_{0}.f'(1)=0 \Rightarrow f'(1)=-2$

Thus; $L(x)=f(1)+f'(1)(x-1)=0+(-2)(x-1)$

$f(1.01)\approx L(1.01)=(-2)(1.01-1)=-0.02$

5. Estimate $\arctan(1.01)-\arctan(0.99)$ using linear approximation.

Solution:

The points 1.01 and 0.99 are close to 1, so the linearization of $f(x)=\arctan x$ about $x=1$ provides linear appeoximation for both $\arctan(1.01)$ and $\arctan(0.99)$

$f(x)=\arctan x \Rightarrow f(1)=\arctan 1=\frac{\pi}{4}$

$f'(x)=\frac{1}{1+x^2} \Rightarrow f'(1)=\frac{1}{1+1}=\frac{1}{2}$

The linearization of f about $x=1$;

$L(x)=f(1)+f'(1)\,(x-1)=\frac{\pi}{4}+\frac{1}{2}(x-1)$

$\arctan(1.01)\approx L(1.01)=\frac{\pi}{4}+\frac{1}{2}(1.01-1)=\frac{\pi}{4}+\frac{1}{2}(0.01)$

$\arctan(0.99)\approx L(0.99)=\frac{\pi}{4}+\frac{1}{2}(0.99-1)=\frac{\pi}{4}+\frac{1}{2}(-0.01)$

$\arctan(1.01)-\arctan(0.99)\approx \cancel{\frac{\pi}{4}}+\frac{1}{2}(0.01)-\cancel{\frac{\pi}{4}}+\frac{1}{2}(0.01)=0.01$