MATH 117 & MATH 119

RECITATION 3

Chapter 4 : More Applications of Differentiation

4.4 Extreme Values
4.5 Concavity and Inflections
4.6 Sketching the Graph of a Function

4.4 Extreme Values

Absolute Extreme Values: Function f has absolute maximum value $f(x_0)$ at $x_0 \in D(f)$ if $f(x) \leq f(x_0)$ for all $x \in D(f)$, f has absolute minimum value $f(x_1)$ at $x_1 \in D(f)$ if $f(x) \geq f(x_1)$ for all $x \in D(f)$.

Local Extreme Values: Function f has local maximum value $f(x_0)$ at $x_0 \in D(f)$ if $f(x) \leq f(x_0)$ for all $x \in (x_0-h, x_0+h)$, $h \in \mathbb{R}^+$, f has local minimum value $f(x_1)$ at $x_1 \in D(f)$ if $f(x) \geq f(x_1)$ for all $x \in (x_1-h, x_1+h)$, $h \in \mathbb{R}^+$.

Critical Points, Singular Points and Endpoints

$f(x)$ can have local extreme values only at points x of three special types:

(i) Critical Points of f ($x \in D(f)$, $f'(x) = 0$)
(ii) Singular Points of f ($x \in D(f)$, $f'(x)$ is not defined)
(iii) Endpoints of $D(f)$.

Existence of Extreme Values

(i) If the domain of f is closed and finite interval and f is continuous on that domain, then f must have an absolute maximum value and an absolute minimum value.

(ii) If f is continuous on the open interval (a,b) and $\lim_{x \to a^+} f(x) = L$, $\lim_{x \to b^-} f(x) = M$

If $f(u) > L$ and $f(u) > M$ for some $u \in (a,b)$ f has absolute maximum on (a,b).
If $f(v) < L$ and $f(v) < M$ for some $v \in (a,b)$ f has absolute minimum on (a,b).

The First Derivative Test

Suppose that f is continuous on x_0 and x_0 is not an endpoint of $D(f)$

(i) If there exists an open interval (a,b) containing x_0 such that $f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b) then f has local maximum value at x_0.

(ii) If there exists an open interval (a,b) containing x_0 such that $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b) then f has local minimum value at x_0.

1. Find and classify all critical points of the following functions.

(a) $f(x) = x^2 . e^{-x}$

Solution:

The domain of f is $\mathbb{R}$.

$f'(x) = 2x . e^{-x} - x^2 . e^{-x} = e^{-x}(2x - x^2) = e^{-x} . x(2-x)$

To find critical points ; $f'(x) = 0 \Rightarrow e^{-x} . x . (2-x) = 0$, for any $x \in \mathbb{R}$, $e^{-x} \neq 0$

$x = 0$ OR $x = 2$ are in the domain of f

They are the critical points of f.

To classify the critical points;

x		0		2	
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$	$\smile$ local minimum	$\nearrow$	$\frown$ local maximum	$\searrow$

$f(0)$ is local minimum value

$f(2)$ is local maximum value.

(b) $f(x) = \dfrac{x}{\sqrt[3]{x^2 - 4}}$

Solution:

$x^2 - 4 = (x-2)(x+2) = 0 \Rightarrow x = \pm 2$, so the domain of f is $\mathbb{R} - \{-2, 2\}$.

$$f'(x) = \frac{\sqrt[3]{x^2-4} - x . \frac{1}{3}(x^2-4)^{-2/3} . 2x}{(\sqrt[3]{x^2-4})^2} = \frac{(x^2-4)^{1/3} - \frac{2}{3}x^2(x^2-4)^{-2/3}}{(x^2-4)^{2/3}} \quad \frac{(x^2-4)^{2/3}}{(x^2-4)^{2/3}}$$

$$= \frac{x^2 - 4 - \frac{2}{3}x^2}{(x^2-4)^{4/3}} = \frac{x^2 - 12}{3 . (x^2-4)^{4/3}}$$

To find critical points: $f'(x) = 0 \Rightarrow \dfrac{x^2 - 12}{3(x^2-4)^{4/3}} = 0 \Rightarrow \begin{matrix} x^2 - 12 = 0 \\ (x^2-4)^{4/3} \geq 0 \end{matrix} \Rightarrow x = \pm 2\sqrt{3} \in D(f)$

are the critical points of f

To classify the critical points:

x		$-2\sqrt{3}$		$2\sqrt{3}$	
$f'(x)$	$+$		$-$		$+$
$f(x)$	$\nearrow$	$\frown$ local maximum	$\searrow$	$\smile$ local minimum	$\nearrow$

Thus; $f(-2\sqrt{3})$ is local maximum value and $f(2\sqrt{3})$ is local minimum value.

(c) $f(x) = x^2(x-12)^2$

Solution:

The domain of f is $\mathbb{R}$.

$f'(x) = 2x(x-12)^2 + x^2 \cdot 2(x-12) = (x-12)\left[2x(x-12)+2x^2\right]$

$= (x-12)(4x^2-24x) = 4(x-12)\,x\,(x-6)$

To find the critical values; $f'(x)=0 \Rightarrow 4x(x-6)(x-12)=0$

$x=0,\ x=6,\ x=12$ are in the domain of f.

They are the critical values of f.

To classify the critical values;

x		0		6		12	
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\searrow$	$\smile$	$\nearrow$	$\frown$	$\searrow$	$\smile$	$\nearrow$
		local minimum		local maximum		local minimum	

$f(0)$ and $f(12)$ are local minimum values.

$f(6)$ is local maximum value.

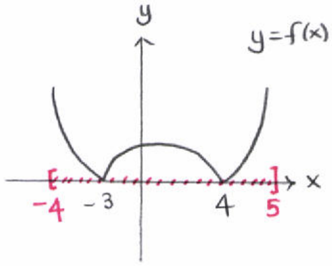
2. For the given functions, first determine and justify whether the function has absolute maximum and minimum. Then find and classify all local and absolute extreme values of the function.

(a) $f(x) = |x^2 - x - 12|$ on $[-4,5]$.

Solution:

Since absolute value function and polynomials are continuous, their composition is also continuous. Thus, f is a continuous function.

f is defined on a finite and closed interval $[-4,5]$ and it is continuous, so it attains its absolute maximum and minimum values on this interval.



$$f(x) = \begin{cases} x^2-x-12 & \text{if } x\in(-\infty,-3]\cup[4,\infty) \\ -x^2+x+12 & \text{if } x\in(-3,4) \end{cases}$$

$$f'(x) = \begin{cases} 2x-1 & \text{if } x\in(-\infty,-3)\cup(4,\infty) \\ -2x+1 & \text{if } x\in(-3,4) \end{cases}$$

$f'(x)$ does not exist at $x=-3$ and $x=4$ on $[-4,5]$ (singular points)

So $f'(x)=0$ when $x = 1/2 \in [-4,5]$ (critical point)

$x=-4,\ x=5$ are the end points

Extreme values occur possibly at critical point $x=1/2$, singular points $x=-3, x=4$, end points $x=-4,\ x=5$.

$f(1/2) = \left|\frac{1}{4} - \frac{1}{2} - 12\right| = \frac{49}{4}$

$f(-3)=0$

$f(4)=0$

$f(-4)=8$

$f(5)=8$

Thus; f has absolute maximum at $\left(\frac{1}{2}, \frac{49}{4}\right)$ and absolute minimum at $(-3,0)$, $(4,0)$.