MATH 117 & MATH 119

RECITATION 2

Chapter 4 : More Applications of Differentiation
4.1 Related Rates
4.3 Indeterminate Forms

4.1 Related Rates

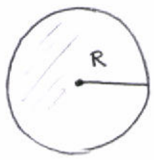
When two or more quantities that change with time are linked by an equation, that equation can be differentiated with respect to time to produce an equation linking the rates of change of the quantities.

How to solve related-rate problems

1. Understand the relation between the variable quantities. What is given? What is to be found?
2. Make a sketch if possible.
3. Define symbols to express given and required quantities and rates.
4. Identify one or more equations linking the variable quantities.
5. Differentiate the equations implicitly with respect to the time, regarding all variable quantities as a function of time.
6. Substitute any given values for the quantities and their rates, then solve the resulting equations for the unknown quantities and rates.

1. The area of a circle is decreasing at a rate of 2 cm^2/min. How fast is the radius of the circle changing when the area is 100 cm^2?

Solution:



The radius and the area of the circle are variable quantities and they can be expressed as functions of time.

$R = R(t)$ radius of the circle
$A = A(t)$ area of the circle

The relation between radius and area: $A = \pi R^2$

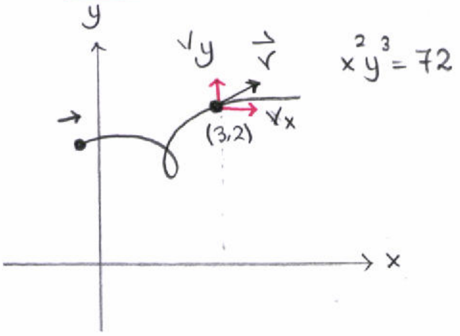
So we have $A(t) = \pi [R(t)]^2 \Rightarrow R(t) = \sqrt{\frac{A(t)}{\pi}}$, $A = 100\ cm^2$, $\frac{dA}{dt} = -2\ cm^2/min$, $\frac{dR}{dt} = ?$

Differentiate with respect to t: $\frac{dR}{dt} = \frac{1}{2} \frac{1}{\sqrt{\frac{A(t)}{\pi}}} \cdot \frac{1}{\pi} \frac{dA}{dt}$

$\frac{dR}{dt} = \frac{1}{2\pi} \frac{\sqrt{\pi}}{\sqrt{100}} (-2) = -\frac{1}{10\sqrt{\pi}}$. The radius is decreasing at the rate $\frac{1}{10\sqrt{\pi}}$ cm/min.

2. A point is moving to the right along the first-quadrant portion of the curve $x^2y^3 = 72$. When the point has coordinates $(3,2)$, its horizontal velocity is 2 units/s. What is its vertical velocity?

Solution:



x and y coordinates of the point are variable quantities.

Let $x(t)$ and $y(t)$ be coordinates of the point as functions of time.

$\frac{d}{dt}y(t)$ gives vertical velocity, $\frac{d}{dt}x(t)$ gives horizontal velocity.

The equation of the curve can be written as $[x(t)]^2 . [y(t)]^3 = 72$ and

$x = 3$, $y = 2$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = ?$

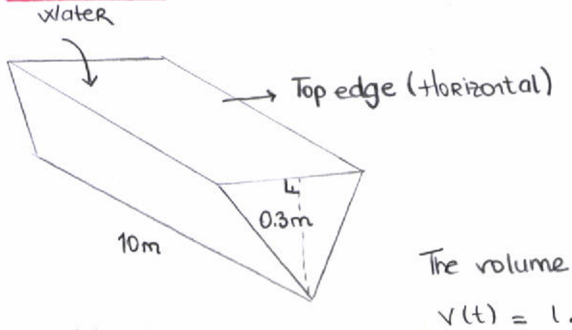
Differentiate the equation

$$2\,x(t) . \frac{dx}{dt} . [y(t)]^3 + 3\,[y(t)]^2\, \frac{dy}{dt} . [x(t)]^2 = 0$$

$$2.3.2\;\; 2^3 + 3.\,2^2 . \frac{dy}{dt}\, 3^2 = 0 \Rightarrow \frac{dy}{dt} = -\frac{2^5 . 3}{2^2\; 3^3} = -\frac{8}{9} \text{ units/s}$$

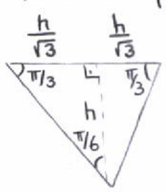
3. The cross section of a water trough is an equilateral triangle with top edge horizontal. If the trough is 10 m long and 30 cm deep, and water is flowing at a rate of 1/4 m^3/min, how fast is the water level changing when the water is 20 cm deep at the deepest?

Solution:



Water trough

The volume of water and the height of water are variable quantities, denoted by $V = V(t)$, $h = h(t)$.



Area of the cross section

$$A(t) = \frac{1}{2} . h(t) . \frac{2h(t)}{\sqrt{3}} = \frac{[h(t)]^2}{\sqrt{3}}$$

The volume of the water:

$$V(t) = l . A(t) = 10 . \frac{[h(t)]^2}{\sqrt{3}}$$

$h = 0.2$

$\frac{dV}{dt} = \frac{1}{4}$ $\quad \frac{dh}{dt} = ?$

Differentiate the equation:

$$\frac{dV}{dt} = \frac{10}{\sqrt{3}} . 2\,h(t)\, \frac{dh}{dt} \Rightarrow \quad \frac{1}{4} = \frac{\cancel{10}}{\sqrt{3}} . 2 . \frac{2}{\cancel{10}} . \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\sqrt{3}}{16}$$

4. If a truck factory employs x workers and has daily operating expenses of \$y, it can produce $P = (1/3)\ x^{0.6}\ y^{0.4}$ trucks per year. How fast are the daily expenses decreasing when they are \$ 10.000 and the number of workers is 40, if the number of workers is increasing at 1 per day and production is remaining constant?

Solution:

We are given $x = 40,\ \frac{dx}{dt} = 1,\ y = 10.000\ .\ \frac{dy}{dt} = ?$

Observe that since the production is constant, $\frac{dP}{dt} = 0$

Differentiate $P = \frac{1}{3} \cdot x^{0.6}\ y^{0.4}$ with respect to t

$$\frac{dP}{dt} = \frac{1}{3} \cdot (0.6)\ x^{-0.4}\ \frac{dx}{dt} \cdot y^{0.4} + \frac{1}{3}\ x^{0.6}\ (0.4)\ y^{-0.6}\ \frac{dy}{dx}$$

$$0 = \cancel{\tfrac{1}{3}}\ (0.6)\ (40)^{-0.4} \cdot 1 \cdot (10000)^{0.4} + \cancel{\tfrac{1}{3}}\ (40)^{0.6} \cdot (0.4)\ (10000)^{-0.6} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{(0.6)\ (40)^{-0.4}\ (10000)^{0.4}}{(0.4)\ (40)^{0.6}\ (10\,000)^{-0.6}} = -375$$

4.3 Indeterminate Forms

Types of Indeterminate Forms: $\frac{0}{0},\ \frac{\infty}{\infty},\ 0.\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$

The First L'Hôpital Rule: Suppose the functions f and g are differentiable on the interval (a,b) and $g'(x) \neq 0$. Suppose also that

(i) $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$

(ii) $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ (where L is finite, ∞ or, $-\infty$)

Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

($x\to a^+$ can be replaced by $x\to b^-$, $a=\infty$ or $b=-\infty$ are allowed)
$x \to c$, $a<c<b$

The Second L'Hôpital Rule: Suppose that f and g are differentiable on (a,b), $g'(x) \neq 0$ Suppose also that

$\lim_{x\to a^+} g(x) = \pm\infty$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ (L is finite, ∞ or $-\infty$)

Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

($x\to a^+$, $x\to b^-$, $x\to c$, $a=\infty$, $b=-\infty$)