Starting from now, we'll try to solve the wave equation (hyperbolic canonical form), the heat equation (parabolic canonical form) and the Laplace equation (elliptic canonical form) with certain initial conditions (IC) and boundary conditions (BC). Whenever we are given a P.D.E with certain initial and/or boundary conditions, a mathematician will always ask whether a solution exists, if exists whether unique or not. The terminology for these kind of questions is called “Well-Posedness”.

**Well-Posed Problems**: Suppose we have a P.D.E on a domain together with a set of initial and/or boundary conditions. We say that this problem is well-posed if it has the following properties:

1) **Existence**: There exists at least one solution satisfying all conditions.
2) **Uniqueness**: There is at most one solution i.e. the solution is unique.
3) **Stability**: The unique solution depends continuously on the initial and/or boundary data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little. (This requires the concept of “distance” or metric.)

**Note**: The stability property is normally required in physical problems since the data is never precise, there may always be small perturbations. Hence, any little change in data shouldn't change the solution significantly if the problem is stable.
Example: (Not a Well-Posed Problem)

Consider the Laplace equation

$$u_{xx} + u_{yy} = 0$$

with initial conditions on $y = 0$:

1. $u(x,0) = 0$, $u_y(x,0) = 0$

We have $u(x,y) = 0$ as a solution to this problem. Now, let's change the initial conditions a little on $y = 0$.

2. $u(x,0) = 0$, $u_y(x,0) = \frac{\sin(nx)}{n}$

As we can see $\sup_{x \in \mathbb{R}} \left\{ \frac{\sin(nx)}{n} \right\} \to 0$ as $n \to \infty$. Hence, the initial data of the second case can be made arbitrarily close to the data of the first case by choosing $n$ large enough.

It can be easily verified that $u(x,y) = \frac{\sinh(ny) \cdot \sin(nx)}{n^2}$ is the solution to the second problem. But, as $n$ increases ($n \to \infty$) $u(x,y) = \frac{\sinh(ny) \cdot \sin(nx)}{n^2}$ doesn't get small i.e. it doesn't get close to $u(x,y) = 0$ since $\lim_{n \to \infty} \frac{\sinh(ny)}{n^2} = \lim_{n \to \infty} \frac{e^{ny} - e^{-ny}}{n^2} = +\infty$.

Hence, even the initial data are close to each other for large $n$, the solutions are not. This shows that the problem is not stable, so it's not Well-Posed.

However, we will see that it's not the case for the wave and the heat equations.
**WAVE EQUATION**

We'll consider one dimensional wave equation.

\[ U_{tt} - c^2 U_{xx} = F(x,t) \]

for an infinite uniform string of small cross-section along x-axis moving in vertical direction.

For a fixed time \( t \):

- \( U(x,t) \equiv \) the displacement of the string at point \( x \) at time \( t \).
- \( F(x,t) \equiv \) some external force.
- \( c^2 \equiv \) constant depending of the material of the string.

**Homogeneous Wave Equation:** \( (F(x,t) = 0) \)

\[ U_{tt} - c^2 U_{xx} = 0 \quad (*) \]

By the method of characteristics \( (a=1, b=0, c=-c^2, \Delta = t^2 - 2c) \) we get \( \frac{dx}{dt} = \pm \frac{2c}{2} = \mp c \) i.e \( dx + cdt = 0 \) or \( dx - cdt = 0 \)

and eventually \( x + ct = c_1 \) & \( x - ct = c_2 \). By choosing \( \xi = x + ct, \eta = x - ct \) we obtain

\[ U_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}, \quad U_{tt} = c^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}) \]

Substitution of these into (*) gives: \( -4c^2 U_{\xi\eta} = 0 \)

Since \( c \neq 0 \) we have \( U_{\xi\eta} = 0 \)

By integrating with respect to \( \eta \):

\[ U_\xi = f(\xi) \]

Then, \( \xi \equiv U(\xi,\eta) = F(\xi) + G(\eta) \)

Hence, we get the general solution: \( U(x,t) = F(x+ct) + G(x-ct) \)
Cauchy Problem (Infinite String)

\[ U_{tt} - c^2 U_{xx} = 0 \quad x \in \mathbb{R}, \quad t > 0 \]

Initial Conditions
\[
\begin{align*}
U(x, 0) &= f(x) & x \in \mathbb{R} \\
U_t(x, 0) &= g(x) & x \in \mathbb{R}
\end{align*}
\]

We know the general solution, \( U(x, t) = F(x + ct) + G(x - ct) \)

By applying the initial conditions
\[
\begin{align*}
U(x, 0) &= f(x) = F(x) + G(x) \\
U_t(x, 0) &= g(x) = c \cdot F'(x) - c \cdot G'(x) = c \cdot (F'(x) - G'(x))
\end{align*}
\] (1)

By integrating the second we obtain
\[
F(x) + G(x) = \int_{0}^{x} f(s) ds + A
\]

So,
\[
\begin{align*}
F(x) &= \frac{1}{2} f(x) + \frac{1}{2c} \int_{0}^{x} g(s) ds + \frac{A}{2} \\
G(x) &= \frac{1}{2} f(x) - \frac{1}{2c} \int_{0}^{x} g(s) ds - \frac{A}{2}
\end{align*}
\]

\[
U(x, t) = F(x + ct) + G(x - ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{0}^{x+ct} g(s) ds + \frac{A}{2}
\]
\[
+ \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_{0}^{x-ct} g(s) ds - \frac{A}{2}
\]

\[
U(x, t) = \frac{1}{2} \left[ f(x + ct) + f(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds
\]

\[ \text{d’Alembert Formula (or Solution)} \]
Example: Find the solution of the Cauchy problem
\[ \begin{align*}
    u_{tt} - 4u_{xx} &= 0 \quad t > 0, \quad x \in \mathbb{R} \\
    u(x, 0) &= \sin x \quad x \in \mathbb{R} \\
    u_t(x, 0) &= \cos x \quad x \in \mathbb{R}
\end{align*} \]

Solution: By d'Alambert's formula
\[
    u(x, t) = \frac{1}{2} \left( \sin(x + 2t) + \sin(x - 2t) \right) + \frac{1}{2} \int_{x-2t}^{x+2t} \cos(s) \, ds
\]
\[
    = \frac{1}{2} \left[ \sin(x + 2t) + \sin(x - 2t) \right] + \frac{1}{4} \left[ \sin(x + 2t) - \sin(x - 2t) \right]
\]
\[
    = \frac{3}{4} \sin(x + 2t) + \frac{1}{4} \sin(x - 2t)
\]

Remarks: 1) \(u(x,t)\) is twice continuously differentiable if
\(f\) is twice continuously differentiable and \(g\) is continuously differentiable by d'Alambert's formula.

2) Cauchy problem is well-posed.
   i) Solution exists. (by d'Alambert's formula.)
   ii) The solution is unique. (The equation is linear and we know the general solution. The solution to the Cauchy problem is uniquely determined by initial conditions \(f, g\), i.e. \(F \& G\) in the general solution are uniquely determined by \(f, g\).
   iii) The solution is stable i.e. depends continuously on the initial data. A small change in either \(f\) or \(g\) results in a small change in \(u(x,t)\).
Now, we'll show stability mathematically: We show that for any finite time interval $0 \leq t \leq T$, $u(x,t)$ continuously depends on initial data, i.e., for every $\varepsilon > 0$, there exists $\delta(\varepsilon, T)$ such that $\sup_{t \in [0,T]} |f_1(t) - f_2(t)| < \delta$ and $\sup_{t \in [0,T]} |g_1(t) - g_2(t)| < \delta$ imply that

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |u_1(x,t) - u_2(x,t)| < \varepsilon.$$ 

Here $f_1$ and $g_1$ are the initial conditions of the first Cauchy problem with the unique solution $u_1(x,t)$, $f_2$ and $g_2$ are the initial conditions of the second Cauchy problem with the unique solution $u_2(x,t)$.

Now,

$$u_1(x,t) = \frac{1}{2} \left( f_1(x+ct) + f_1(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g_1(s) \, ds,$$

$$u_2(x,t) = \frac{1}{2} \left( f_2(x+ct) + f_2(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g_2(s) \, ds.$$

Hence,

$$|u_1 - u_2| \leq \frac{1}{2} \left| f_1(x+ct) - f_2(x+ct) \right| + \frac{1}{2} \left| f_1(x-ct) - f_2(x-ct) \right| + \frac{1}{2c} \int_{x-ct}^{x+ct} |g_1(s) - g_2(s)| \, ds$$

$$\leq \frac{1}{2} \delta + \frac{1}{2} \delta + \frac{1}{2c} \int_{x-ct}^{x+ct} \delta \, ds = \delta + \frac{\delta \cdot (2cT)}{2c} \leq \delta + \delta \cdot T \leq \delta \cdot (1+T).$$

Hence, given $\varepsilon > 0$, if we choose $\delta < \frac{\varepsilon}{1+T}$, then we get what we wanted.
Interpretation of Solution

Firstly, you should note that \( u(x,0) = f(x) \) gives the initial position of point \( x \), \( u_t(x,0) = g(x) \) gives the initial velocity of point \( x \). (You could think a string as a union of points.)

Let's consider the following special case:

\[
U_{tt} - c^2 U_{xx} = 0, \quad U(x,0) = f(x), \quad U_t(x,0) = 0 \quad (\text{No initial velocity})
\]

Then \( U(x,t) = \frac{f(x+ct) + f(x-ct)}{2} \) as the graph of \( f(x+ct) \) is the graph of \( f(x) \) shifted right (+) / left (-) \( ct \) units.

\( U \) is the sum of two waves (graph of \( f \) can be considered as a wave)

where \( f(x+ct) \) moves right, \( f(x-ct) \) moves left with velocity \( c \).

**Example:**

\[
f(x) = \begin{cases} 
  x+1 & \text{if } -1 < x \leq 0 \\
  1-x & \text{if } 0 < x < 1 \\
  0 & \text{otherwise}
\end{cases}
\]

Let's draw \( \frac{f(x+ct) + f(x-ct)}{2} \) for \( t = 0, \ t = \frac{1}{2c}, \ t = \frac{1}{c}, \ t = \frac{2}{c} \)
In general, \((u_{tt} - c^2 u_{xx} = 0), \ u(x,0) = f(x), \ u_t(x,0) = g(x)\).

\[
U(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(s)ds + \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(s)ds
\]

which is the sum of the waves \(F\) & \(G\) where \(F\) propagates left with the speed \(c\) and \(G\) propagates right with the same speed \(c\).

**Domain of Dependence & Domain (or Region) of Influence**

Question: What do we need to know in order to calculate the solution \(U(x,t)\) at a point \((x_0,t_0)\)?

By d'Alembert formula,

\[
U(x_0,t_0) = \frac{1}{2} f(x_0+ct_0) + f(x_0-ct_0) + \frac{1}{2c} \int_{x_0-ct_0}^{x_0+ct_0} g(s)ds
\]

Hence, knowing \(g\) on the interval \([x_0-ct_0, x_0+ct_0]\) and \(f\) at the boundary points of this interval is enough to compute \(U(x_0,t_0)\). We call \([x_0-ct_0, x_0+ct_0]\) the domain of dependence of \((x_0,t_0)\). Actually, if we consider the characteristics through \((x_0,t_0)\), the intersections of them with the \(x\)-axis will give the boundaries of the domain of dependence. The characteristic triangle formed is called the characteristic triangle.
Question: What happens to the solution if the initial data changes on some interval \([a, b]\) on the x-axis?

\[ u(x_0, t_0) \]

Changes if the domain of the dependence \([x_0 - ct_0, x_0 + ct_0]\) intersects \([a, b]\).

Hence, for the points in II & III region above, the domain of the dependence doesn't intersect \([a, b]\) at all, and their values don't change.

For the points in region I, where \( I = \{ (x,t) | x - ct \leq b \text{ and } x + ct \geq a \} \)

the domains of dependence intersects \([a, b]\), so their values change. Here \( I \) is called the domain of influence.

Let's see these on a simple example.

\[ \text{Ex: } u_{tt} - \frac{1}{4} u_{xx} = 0 \quad u(x, 0) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad u_t(x, 0) = 0 \]

Here, \( f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = 0 \). So, there's an initial displacement for \(-1 \leq x \leq 1\) only and initial velocity is zero everywhere. (We can consider \( f \) as change on \([1, 1]\) from everywhere zero function.

\[ u(x, t) = \frac{f(x + \frac{1}{2}t) + f(x - \frac{1}{2}t)}{2}, \quad x + \frac{1}{2}t = 1 \]

On B \( u(x, t) \equiv 0 \) since domains of dependence don't intersect \([1, 1]\) where \( f \) is non-zero.

On A \( u(x, t) \) won't be equal to the identical zero function.
Actually, we can divide $A$ into further parts to find $u(x,t)$.

On $A_1$: $u(x,t) = \frac{1+1}{2} = 1$

On $A_2$: $u(x,t) = \frac{1+0}{2} = \frac{1}{2}$

On $A_3$: $u(x,t) = \frac{0+0}{2} = 0$

Before we continue the wave equation on the half line i.e. semi-infinite string, let's consider the general wave eqn. with constant coefficients:

\[ uu_{tt} + b u_{xt} + c u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad a, b, c \in \mathbb{R} \]

Assume $b^2 - 4ac > 0$ (Hyperbolic). Then, we get the characteristics by solving $\frac{dx}{dt} = \lambda_{1,2}$ where $\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

which are $dx - \lambda_1 dt = 0$, $dx - \lambda_2 dt = 0$

$x - \lambda_1 t = C_1$, $x - \lambda_2 t = C_2$

By choosing $\xi = x - \lambda_1 t$ and $\eta = x - \lambda_2 t$, we reduce the eqn to $u_{\xi \eta} = 0$

so we get the general solution, $u(x,t) = F(x-\lambda_1 t) + G(x-\lambda_2 t)$

**Wave Equation on the Half Line (Semi-infinite String)**

\[ uu_{tt} - c^2 u_{xx} = 0 \quad x > 0, \quad t > 0 \]

\[ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{Initial Conditions} \]

\[ x > 0 \]

\[ U(0,t) = 0, \quad t > 0 \quad \text{Boundary Condition} \]

(Boundary condition gives that one end of string is fixed.)
We'll try to use d’Alembert's formula

\[ U(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds \]

But, it's defined only for \( x-ct > 0 \). To find the solution \( U(x,t) \) for \( (x,t) \) with \( x-ct < 0 \), we'll use the boundary condition.

Let's remember the general solution to the homogeneous wave equation

\[ U(x,t) = F(x+ct) + G(x+ct). \]

(\( G \) is not determined when \( x-ct < 0 \))

But, by boundary condition

\( U(0,t) = F(ct) + G(-ct) = 0 \),

we get \( G(-ct) = -F(ct) \). If we set \( z = -ct \), then

\( G(z) = -F(-z), \quad z < 0 \) (Remember: \( F(x) = \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(s)\,ds + A/2 \))

Replacing \( z \) by \( x-ct \) we obtain \( G(x-ct) = -F(ct-x) \)

So, \( G(x-ct) = -\frac{f(ct-x)}{2} - \frac{1}{2c} \int_0^{ct-x} g(s)\,ds - \frac{A}{2} \)

and

\( F(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c} \int_0^{x+ct} g(s)\,ds + \frac{A}{2} \) for \( x-ct < 0 \)

Hence, \( U(x,t) = \)

\[
\begin{cases} 
\frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds & \text{if } x-ct > 0 \\
\frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s)\,ds & \text{if } x-ct < 0 
\end{cases}
\]
Geometrically, what does this mean?

To find the domain of dependence we use characteristics
(1) \( x-ct = x_0 - ct_0 \) for \((x_0, t_0)\)
(2) \( x+ct = x_0 + ct_0 \)

But when (1) characteristic hits boundary \( x=0 \), it's reflected and we get a new characteristic \( x+ct = ct_0 - x_0 \) which determines the left point of the domain of dependence of \((x_0, t_0)\).

Example: Determine the solution of
\[ u_{tt} = 4u_{xx} \quad x>0, \, t>0 \]
\[ u(x,0) = |\sin x| \quad x>0, \, u_t(x,0) = 0 \quad x>0 \]
\[ u(0,t) = 0 \quad t>0 \]

Solution:
\[ x>ct \quad u(x,t) = \frac{1}{2} (|\sin(x+2t)| + |\sin(x-2t)|) \]
\[ x<ct \quad u(x,t) = \frac{1}{2} |\sin(x+2t)| - |\sin(2t-x)| \]

Note that \( u(x,t) \) is continuous along \( x-ct=0 \).

Remark: As before, in this case for the existence of solution, \( f \) must be twice continuously differentiable, and \( g \) must be continuously differentiable and in addition \( f(0) = g(0) = f''(0) = 0 \).

Exercise: (Semi-infinite String with a Free End)
Determine the solution of
\[ u_{tt} - c^2 u_{xx} = 0 \quad x>0, \, t>0 \]
\[ u(x,0) = f(x) \quad x>0 \] \[ \text{Initial Conditions} \]
\[ u_t(x,0) = g(x) \quad x>0 \]
\[ u_x(0,t) = 0 \quad t>0 \] \[ \text{Boundary Condition} \]
Non-Homogeneous Wave Equation (Waves with a Source)

We'll consider: \( u_{tt} - c^2 u_{xx} = h(x,t), \quad x \in \mathbb{R}, \quad t > 0 \)
\[ u(x,0) = f(x), \quad x \in \mathbb{R} \]
\[ u_t(x,0) = g(x), \quad x \in \mathbb{R}. \]

Here \( h(x,t) \) can be considered as an external force acting on an infinite string.

Since our equation is linear, we can write the general solution as sum of two solutions (like homogeneous and particular solutions in O.D.E) coming from two problems

1) \( v_{tt} - c^2 v_{xx} = 0, \quad v(x,0) = f(x), \quad v_t(x,0) = g(x) \)

Homogeneous Wave Eqn. with Non-Zero Initial Conditions

2) \( w_{tt} - c^2 w_{xx} = h(x,t) \quad w(x,0) = 0, \quad w_t(x,0) = 0 \)

Non-Homogeneous Wave Eqn. with Zero Initial Conditions

So, \( u(x,t) = v(x,t) + w(x,t) \).

By d'Alambert+ Formula we know that \( v(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \).

So, it's all about finding \( w(x,t) \).

Proposition: \( w(x,t) = \frac{1}{2c} \iint_{\Delta} h \cdot dA \) where \( \Delta \) is the characteristic triangle of \((x,t)\) i.e.

Proof: There are a couple of proofs of this, but we'll use the one which uses Green's Theorem.

Recall: For any \( P, Q \) (cont'd diff'ble on \( \Omega \subset \mathbb{R}^2 \))
\[ \int_{\partial \Omega} P \, dt + Q \, dx = \iint_{\Omega} (P_x - Q_t) \, dA \] by Green's Thm.
Now let's integrate $W_{tt} - c^2 W_{xx} = h(x,t)$ on $\Delta$

$$\iint h \, dx \, dt = \iint (W_{tt} - c^2 W_{xx}) \, dx \, dt$$

We can use Green's thm to left hand side

$$\iint (W_{tt} - c^2 W_{xx}) \, dx \, dt = \int_{L_0 + L_1 + L_2} -c^2 W_x \, dt - W_t \, dx$$

We'll evaluate each piece:

1) On $L_0$, i.e. $(x,t) = (x,0)$, $dt = 0$, $W_t(x,0) = 0$

$$\int_{x_o - c t_o}^{x_o + c t_o} \left( -c^2 W_x \, dt - W_t \, dx \right) = \int_{x - c t_o}^{0} 0 \, dx = 0$$

2) On $L_1$, $x + ct = x_0 + ct_0$, so $dx + c \, dt = 0$, i.e. $dt = -\frac{1}{c} \, dx$

$$\int_{L_1} -c^2 W_x \, dt - W_t \, dx = \int_{L_1} -c^2 W_x \left(-\frac{1}{c} \, dx\right) - W_t \left(-\frac{1}{c}\right) \, dt = \int_{L_1} W_x \, dx + W_t \, dt = C \int_{L_1} \, dx$$

$$= C \left( W(x_0, t_0) - W(x_0 + ct_0, 0) \right) = C \cdot W(x_0, t_0)$$

3) On $L_2$, $x - ct = x_0 - ct_0$, so $dx - c \, dt = 0$, i.e. $dt = \frac{1}{c} \, dx$. Similar to (2) we get

$$\int_{L_2} -c^2 W_x \, dt - W_t \, dx = -C \int_{L_2} \, dx = -C \left( W(x_0 - ct_0) - W(x_0, t_0) \right) = C \cdot W(x_0, t_0)$$

By adding these three integrals:

$$\iint h \, dx \, dt = 0 + 2C \cdot W(x_0, t_0)$$

Hence,

$$W(x_0, t_0) = \frac{1}{2C} \iint h \, dx \, dt$$
Example: Determine the solution of

\[ u_{tt} - u_{xx} = 1 \]

\[ u(x,0) = \cos x \]

\[ u_t(x,0) = x \]

**Soln:** Here \( c = 1 \), so the characteristic curves are \( x + t = x_0 + t_0 \) and \( x - t = x_0 - t_0 \). If we consider \( U = V + W \) as in the proof,

\[ V(x,t) = \frac{\cos(x-t) + \cos(x+t) + \frac{1}{2.1} \int_{x-t}^{x+t} s \, ds}{2} \]

\[ = \frac{\cos(x-t) + \cos(x+t)}{2} + \frac{1}{2} \left( \frac{(x+t)^2}{2} - \frac{(x-t)^2}{2} \right) \]

\[ = \frac{\cos(x-t) + \cos(x+t)}{2} + \frac{1}{4} \cdot 2x \cdot 2t \]

\[ W(x,t) = \frac{1}{2.1} \int \int_D 1 \, d\Delta = \frac{1}{2} \cdot \frac{(x+t-x+t) \cdot t}{2} \]

\[ = \frac{t^2}{2} \]

\[ U(x,t) = \frac{\cos(x-t) + \cos(x+t)}{2} + x \cdot t + \frac{t^2}{2} \]

Example: \( u_{tt} - 4u_{xx} = e^x \)

\[ u(x,0) = x \]

\[ u_t(x,0) = 1 \]

**Initial Conditions**

\[ u(0,t) = 0 \]

**Boundary Condition**

**Soln:** Again we can separate it into two problems for \( V, W \) such that \( U = V + W \)
First, \[ V_{tt} - 4 V_{xx} = 0 \quad \text{and} \quad W_{tt} - 4 W_{xx} = e^x \]

\[ \begin{align*}
V(x,0) &= x \\
V_t(x,0) &= 1 \\
V(0,t) &= 0 \\
W(x,0) &= 0 \\
W_t(x,0) &= 0 \\
W(0,t) &= 0
\end{align*} \]

It's the semi-infinite string with a fixed end, so as we did before

\[ V(x,t) = \begin{cases} 
\frac{x+2t + x-2t}{2} + \frac{1}{2.2} \int_{x-2t}^{x+2t} 1 \, ds & \text{if } x > 2t \\
\frac{(x+2t) - (2t-x)}{2} + \frac{1}{2.2} \int_{x+2t}^{x+2t} 1 \, ds & \text{if } x < 2t
\end{cases} \]

Similarly, when \( x > 2t \), \[ W(x,t) = \frac{1}{2.2} \int_{x-2t}^{x+2t} e^x \, dA \]

But, when \( x < 2t \)

\[ W(x,t) = \frac{1}{2.2} \int_{x+2t-2t}^{x+2t+2t} \int_{2t-x}^{x+2t} e^x \, dA \]

Note that when \( (x,t) = (0,t) \), then \( D \) will be just a line segment and the double integral is equal to zero. Hence, \( w(0,t) = 0 \) is automatically satisfied.

Let me compute \( x > 2t \) case and leave \( x < 2t \) to you.

\[ W(x,t) = \frac{1}{4} \int_{0}^{x+2t-2t} \int_{x-2t+2t}^{x+2t-2t} e^x \, dx \, dt = \frac{1}{4} \int_{0}^{x+2t-2t} \left( e^{x+2t} - e^{x-2t} \right) \, dt \]

\[ = \frac{e}{4} \left( \frac{e^{x+2t} - e^{x+2t} - e^{-2t} + e^{-2t}}{2} \right)_{0}^{t} = \frac{e}{4} \cdot \frac{1}{2} \left( e^{2t} + e^{2t} + 1 \right) \]
Now, we'll show that non-homogeneous wave equation with zero initial conditions is well-posed.

\[ u_{tt} - C^2 u_{xx} = h(x,t) \quad u(x,0) = 0 = u_t(x,0) \]

1) **Existence:**

\[ u(x,t) = \frac{1}{2c} \int_{\Delta} h(x,t) \, dA \]

2) **Uniqueness:** Let \( u_1, u_2 \) be two solutions. Then, \( V = u_1 - u_2 \) satisfies \( V_{tt} - C^2 V_{xx} = 0 \), \( V(x,0) = 0 = V_t(x,0) \) which has a unique soln. as proved earlier. Obviously, that unique solution is \( V(x,t) = 0 \) which proves \( u_1 = u_2 \).

3) **Stability:** We'll prove that if \( h(x,t) \) is small, then \( u(x,t) \) is small for \( t \in [0,T] \) which is equivalent to the statement that if \( h(x,t) \) changes little, then \( u(x,t) \) changes little, too.

Now,

\[ |u(x,t)| = \frac{1}{2c} \int_{\Delta} |h(x,t)| \, dA \]

\[ \forall \varepsilon > 0, \text{ choose } S < \frac{2\varepsilon}{T^2} \text{ such that if } \sup_{x \in \mathbb{R}} |h(x,t)| < S, \text{ then } \]

\[ \sup_{t \in [0,T]} |u(x,t)| < \varepsilon \]

\[ \frac{1}{2c} \int_{\Delta} |h(x,t)| \, dA \leq \frac{1}{2c} \int_{\Delta} S \, dA = \frac{S}{2c} \text{ Area}(\Delta) \]

\[ = \frac{S}{2c} \left( \frac{(x+ct - (x-ct)) \cdot t}{2} \right) = \frac{S}{2c} ct^2 \leq \frac{S}{2c} ct^2 \leq \frac{S}{2c} cT^2 = \varepsilon \]

Therefore,

\[ \sup_{t \in [0,T]} |u(x,t)| < \varepsilon \]
THE HEAT (DIFFUSION) EQUATION

Now we'll work on parabolic equations, namely the heat or diffusion equation as its standard model.

\[ U_t \equiv k \cdot U_{xx} \quad \text{\( U_t = k (U_{x1} + \ldots + U_{xn}) \) in higher dimensions} \]

We'll study one dimensional case: \( U(x, t) \) = Temperature at \( x \) at time \( t \).

Here we have a uniform thin rod along \( x \)-axis s.t. all points of a cross-section have the same temperature \( U(x, t) \). If the rod is insulated, then the temperature satisfies \( U_t - k \cdot U_{xx} = 0 \). \( k > 0 \) depends on the material of the rod.

First, we'll consider a finite length rod positioned from 0.

\[ U_t - k \cdot U_{xx} = 0 \quad 0 \leq x \leq L, \quad t > 0 \]

\[ U(x, 0) = f(x) \quad \text{I.C.} \]

\[ U(0, t) = d(t), \quad U(L, t) = \beta(t) \quad \text{B.C.} \]

We'll learn how to solve it later. We'll prove the uniqueness and the stability of the problem first. For this we'll use the maximum principle which works for both parabolic and elliptic equations. (There's an alternative proof by using energy. If you like physics or are interested, you can read in Section 2.3 of P.D.E by Strauss.)

In short, the maximum principle says (without an internal heat source) the hottest or the coldest points happen when \( t = 0 \) or at the end points \( x = 0 \) or \( x = L \) for \( t > 0 \).
Maximum Principle: If \( U(x,t) \) satisfies the heat (diffusion) equation on a rectangle \( \Omega = \{(x,t) \mid 0 \leq x \leq L, 0 \leq t \leq T \} \), then the maximum value of \( U(x,t) \) is assumed either initially \( (t=0) \) or on the lateral sides \( (x=0, x=L) \)

Note: i) To prove the minimum, we can just \( T \) apply the maximum principle to \( -U(x,t) \) which clearly satisfies the heat eqn, too.

ii) We'll use the following notation

\[ B = \{(x,t) \mid t=0, 0 \leq x \leq L, \text{ or } x=0, 0 \leq t \leq T, \text{ or } x=L, 0 \leq t \leq T \} \]

(Blue set in the figure which is claimed to be the place of max.)

Hence, the maximum principle can be stated as:

If \( U(x,t) \in C^2(\Omega) \) satisfies the heat equation \( U_t - kU_{xx} = 0 \), then

\[ \max_{\Omega} U(x,t) = \max_B U(x,t) \]

Proof: Let \( M \) be the maximum value of \( U(x,t) \) on \( B \) which exists due to Extreme Value Theorem (\( U \) is continuous, \( B \) is closed and bounded.) We'll show that \( U(x,t) \leq M \) on \( \Omega \)

Define \( \varphi(x,t) = U(x,t) + \varepsilon x^2 \) for some \( \varepsilon > 0 \). Then,

1. \( \varphi_t - k \varphi_{xx} = U_t - k(U_{xx} + 2\varepsilon) = U_t - kU_{xx} - 2\varepsilon \cdot k = -2\varepsilon \cdot k < 0 \)

2. \( \varphi(x,t) \leq M + \varepsilon \cdot L^2 \) on \( B \).

Suppose \( \varphi(x,t) \) attains its maximum at an interior point \( (x_0,t_0) \) \( 0 < x_0 < L, 0 < t_0 < T \). Then, \( \varphi_t(x_0,t_0) = 0 \) and \( \varphi_{xx}(x_0,t_0) \leq 0 \) (\( \nabla \varphi(x_0,t_0) = 0 \) and Hessian of \( \varphi(x_0,t_0) \) is negative-definite at a maximum point by 2nd Derivative Test.) But, then \( \varphi_t - k \varphi_{xx}(x_0,t_0) \leq 0 \), which is a contradiction to 1 above.
So, there can't be an interior maximum. Suppose \( U(x_{0}, t_{0}) \) has a maximum on top edge, \( 0 < x_{0} < L \), \( t_{0} = T \). Then, 
\[
U_{x}(x_{0}, t_{0}) = 0, \quad U_{xx}(x_{0}, t_{0}) \leq 0 \text{ as before. But, } U_{t}(x_{0}, t_{0}) > 0\] 
as one-sided derivative since 
\[
\lim_{h \to 0^{-}} \frac{U(x_{0}, t_{0}+h) - U(x_{0}, t_{0})}{h} > 0.
\] 
and we get \( U_{t} - kU_{xx} \leq 0 \), a contradiction to \( 1 \).

Since \( U(x_{0}, t_{0}) \) is continuous on closed and bounded domain \( \Omega \), it has the maximum value and by the discussion above \( U(x_{0}, t_{0}) \) attains its maximum value on \( B \), and by \( 2 \) \( U(x_{0}, t_{0}) \leq M + \varepsilon L^{2} \) on \( \Omega \). Hence, \( U(x_{0}, t_{0}) + \varepsilon x^{2} \leq M + \varepsilon L^{2} \) on \( \Omega \) for any \( \varepsilon > 0 \)
\[
U(x_{0}, t_{0}) \leq M + \varepsilon (L^{2} - x^{2}) \] 
on \( \Omega \)

Since it's true for any \( \varepsilon > 0 \), we get \( U(x_{0}, t_{0}) \leq M \) on \( \Omega \).
So, \( \max \ U(x_{0}, t_{0}) = \max \ U(x_{0}, t_{0}) \)
\( \Omega \)
\( B \)

Furthermore, a similar discussion for \( U(x_{0}, t_{0}) \) like one for \( U(x_{0}, t_{0}) \) shows that an interior point can't give the maximum value.

**Example:** Find the maximum and the minimum values of 
\[
U(x_{0}, t_{0}) = (4 - x^{2}) - 4t \] 
on \( 0 \leq x \leq 2, \quad 0 \leq t \leq 1 \)

**proof:** We'll use the maximum principle. It's easy to see that \( U(x_{0}, t_{0}) \) satisfies \( U_{t} - 2U_{xx} = 0 \). So we'll just check the set \( B \)
\( \Omega \)
\( t = 0 \) \( U(x_{0}, t_{0}) = 4 - x^{2} \), \( 0 \leq x \leq 2 \) \( \Rightarrow \) max. value is 4, min. value is 0
\( x = 0 \) \( U(x_{0}, t_{0}) = 4 - 4t \), \( 0 \leq t \leq 1 \) \( \Rightarrow \) 4 0
\( x = 2 \) \( U(x_{0}, t_{0}) = -4t \), \( 0 \leq t \leq 1 \) \( \Rightarrow \) 0 -4
Hence, the max. value of $u(x,t)$ is 4 
the min. value of $u(x,t)$ is $-4$ on $0 \leq x \leq 2, 0 \leq t \leq 1$ 

Now, we'll use the maximum principle to show the uniqueness and the stability of initial/boundary value problem of the heat equation.

$$u_t - ku_{xx} = h(x,t) \quad 0 < x < L, \quad t > 0$$
$$u(x,0) = f(x)$$
$$u(0,t) = \alpha(t), \quad u(L,t) = \beta(t)$$

This problem is also called "Dirichlet problem".

**Uniqueness**: Let $u_1, u_2$ be two solutions. Then, consider

$V = u_1 - u_2$. $V$ will satisfy $V_t - kV_{xx} = 0$, $V(x,0) = 0$, $V(0,t) = V(L,t) = 0$

Then for any $T > 0$, $\max V = \max V$ by the maximum principle:

But, $V \equiv 0$ on $B$. So, $V \leq 0$. Similarly, $-V \leq 0$ (choose $V = u_2 - u_1$)

Hence, $u \equiv 0$ on $-\Omega$ i.e $u_1 - u_2 = 0$, $u_1 = u_2$ on $-\Omega$.

**Stability**: Let $u_1(x,t)$ and $u_2(x,t)$ be the solutions of the problems:

$$u_t - ku_{xx} = h_1(x,t) \quad \quad \quad \quad \quad u_t - ku_{xx} = h_2(x,t)$$
$$u(x,0) = f_1(x) \quad \quad \quad \quad \quad u(x,0) = f_2(x)$$
$$u(0,t) = \alpha_1(t), \quad u(L,t) = \beta_1(t) \quad \quad \quad \quad \quad u(0,t) = \alpha_2(t), \quad u(L,t) = \beta_2(t)$$

in the region $0 \leq x \leq L, \quad t > 0$, respectively.

**Case I**: $h_1(x,t) = h_2(x,t)$. Let $\theta(x,t) = u_1(x,t) - u_2(x,t)$. Then

$\theta_t - k\theta_{xx} = 0$ and
\[ |\vartheta(x,0)| = |f_1(x) - f_2(x)| \quad 0 \leq x \leq L \]
\[ |\vartheta(0,t)| = |d_1(t) - d_2(t)| \quad 0 \leq t \leq T \]
\[ |\vartheta(L,t)| = |\beta_1(t) - \beta_2(t)| \quad 0 \leq t \leq T \]

If \( |f_1(x) - f_2(x)| \leq \varepsilon, \ |d_1(t) - d_2(t)| \leq \varepsilon, \ |\beta_1(t) - \beta_2(t)| \leq \varepsilon \), then by the maximum principle \( \max_{\Omega} |\vartheta(x, t)| = \max_{\Omega} |\vartheta(t)| \leq \varepsilon \)

So, \( |\vartheta(x, t)| = |u_1(x, t) - u_2(x, t)| \leq \varepsilon \).

**Case II:** Let \( |h_1(x, t) - h_2(x, t)| < \varepsilon \). Then, \( \vartheta(x, t) = u_1(x, t) - u_2(x, t) \) is the solution to \( \vartheta_t - k \vartheta_{xx} = h_1(x, t) - h_2(x, t) \quad (*) \)
\( \vartheta(x, 0) = f_1(x) - f_2(x) \)
\( \vartheta(0, t) = d_1(t) - d_2(t), \ \vartheta(L, t) = \beta_1(t) - \beta_2(t) \).

Let \( M = \max_{\Omega} \vartheta(x, t) \). Consider \( \tilde{\vartheta}(x, t) = \vartheta(x, t) - (M + \varepsilon t) \). Then, \( \tilde{\vartheta}_t - k \tilde{\vartheta}_{xx} = \frac{\vartheta_t - k \vartheta_{xx} - \varepsilon}{(*)} < 0 \quad \forall (x, t) \in \Omega \). This actually allows us to use the maximum principle (It's similar to (1) in the proof of the maximum principle).
\[ \max_{\Omega} \tilde{\vartheta} = \max_{\Omega} \vartheta \leq \max_{\Omega} (\vartheta - M) + \max_{\Omega} (-M - \varepsilon t) \leq 0 \]
So, \( \tilde{\vartheta}(x, t) = \vartheta(x, t) - (M + \varepsilon t) \leq 0 \) i.e. \( \vartheta(x, t) \leq M + \varepsilon t \) on \( \Omega \).

Using \( -\vartheta(x, t) \) in a similar way, we get \( |\vartheta(x, t)| \leq M + \varepsilon t \)

Hence, if \( |f_1(x) - f_2(x)| < \varepsilon, \ |d_1(t) - d_2(t)| < \varepsilon, \ |\beta_1(t) - \beta_2(t)| < \varepsilon \)
\( x \in [0, L], \quad t \in [0, T] \)
\( t \in [0, T] \)
\( t \in [0, T] \)
\[ |\vartheta(x, t)| = |u_1(x, t) - u_2(x, t)| < \varepsilon + \varepsilon \cdot T = (1 + T) \varepsilon \]

M by Case I
Separation of variables (due to Fourier) is a powerful technique to solve linear PDE's. (It's applicable only for problems with an appropriate symmetry shared by the domain and the equation. In most cases, the domain should be bounded.) Let's outline the main steps:

1. We search for the solutions of the homogeneous linear PDE in the product (or separated) form i.e. \( U(x,t) = X(x)T(t) \)
   (In general, these solutions should satisfy additional conditions coming from (homogeneous) boundary conditions.

2. Separation of variables reduces the problem to two linear ODE's, one for \( X(x) \), one for \( T(t) \), which are easily derived from PDE.

3. These ODE's are solved using boundary conditions.

4. We use a generalization of superposition principle (which means for a linear hom. eqn., a sum of solutions is a solution) to generate a more general solution of the PDE in the form of an infinite series of product solutions.

5. We compute the coefficients of the series by using initial conditions.

**Examples:** Determine whether the following PDE's can be reduced to a pair of ODE's using the separation of variables:

i) \( U_t + U_{xt} + t U_{xx} = 0 \)

ii) \( U_{xx} + (x-y) U_{yy} = 0 \)

1. Assume \( U(x,t) = X(x)T(t) \) and substitute into the eqn.

   \[
   X \cdot T' + X'T' + t X''T = 0 \Longleftrightarrow T'(X + X') = -t X''.T
   \]

   \[
   \Rightarrow \frac{T'}{-tT} = \frac{X''}{X + X'} \quad \text{(Variables are separated, and ratios must be equal to a constant since one side depends on } x \text{ only, the other side depends on } t \text{ only.)}
   \]
Hence, \[ \frac{X''}{X + X'} = - \frac{T'}{t \cdot T} = -\lambda \] (constant), and we get

1) \( X'' + \lambda (X + X') = 0 \)

2) \( T' - \lambda t \cdot T = 0 \)

ii) Assume \( U(x,y) = X(x) \cdot Y(y) \) and substitute into the eqn.

\[ X'' \cdot Y + (x-y) \cdot X \cdot Y'' = 0 \iff X'' \cdot Y = (y-x) \cdot X \cdot Y'' \]

\[ \Rightarrow \frac{X''}{X} = \frac{(y-x) \cdot Y''}{Y} \] which can't be separated.

Now, let's apply this method to find the solution to the heat equation with homogeneous boundary conditions, first.

\[ U_t - k \cdot U_{xx} = 0 \quad 0 < x < L, \quad t > 0 \]

\[ U(x,0) = f(x) \quad 0 \leq x \leq L \quad IC \]

\[ U(0,t) = U(L,t) = 0 \quad t > 0 \quad BC \]

Note: 1) \( k > 0 \) and it's called thermal diffusivity constant.

2) According to boundary conditions (BC), the temperature at the ends (or boundary of the rod) is specified i.e defined by the constant zero function. This type of boundary condition is classified as Dirichlet type BC and the problem is called a Dirichlet problem.

Assume \( U(x,t) = X(x) \cdot T(t) \) and substitute into the heat eqn.

\[ X(x) \cdot T'(t) - k \cdot X''(x) \cdot T(t) = 0 \iff \frac{X''}{X} = \frac{T'}{k \cdot T} = -\lambda \]

We get

1) \( X'' + \lambda X = 0 \)

2) \( T' + \lambda \cdot kT = 0 \)

Now, let's apply the boundary conditions
\[ O = U(0,t) = X(0) T(t), \quad 0 = U(L,t) = X(L) T(t) \]

If \( T(t) = 0 \) for \( t > 0 \), then \( U(x,t) = 0 \), and we get the trivial soln. If \( T(t) \neq 0 \), then \( X(0) = X(L) = 0 \). So, we get Boundary Value Prob.

\[ X'' + \lambda X = 0 \quad X(0) = X(L) = 0 \]

We'll investigate it case by case. First, notice that ODE has characteristic eqn. \( r^2 + \lambda = 0 \)

**Case I:** \( \lambda < 0 \) \( X(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x} \)

Let's substitute boundary cond.

\[ \begin{align*}
0 &= X(0) = c_1 + c_2 \\
0 &= X(L) = c_1 e^{-\sqrt{\lambda}L} + c_2 e^{\sqrt{\lambda}L}
\end{align*} \]

Clearly, coefficient matrix has non-zero determinant, and this implies we have trivial solution \( c_1 = c_2 \). Hence \( X(x) = 0 \), \( U(x,t) = 0 \).

**Case II:** \( \lambda = 0 \) \( X(x) = c_1 + c_2 x \)

\[ \begin{align*}
0 &= X(0) = c_1 \\
0 &= X(L) = c_1 + c_2 L
\end{align*} \]

\( c_1 = c_2 = 0 \) Again, we'll have \( U(x,t) = 0 \).

**Case III:** \( \lambda > 0 \) \( X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \). This time boundary cond.

\[ \begin{align*}
0 &= X(0) = c_1 \\
0 &= X(L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L)
\end{align*} \]

\( c_1 = 0 \) and \( c_2 = 0 \) \( \Rightarrow U(x,t) = 0 \) \( \text{OR} \)

\( c_1 = 0 \) and \( \sin(\sqrt{\lambda}L) = 0 \)

Therefore, we have non-trivial solutions when

\[ \sin(\sqrt{\lambda}L) = 0 \quad \Rightarrow \sqrt{\lambda}L = n\pi, \ n \in \mathbb{Z} \quad \Rightarrow \lambda = \frac{n^2\pi^2}{L}, \ n = 1, 2, 3, \ldots \]

which are \( X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \ldots \)

(Since our equation is homogeneous, any multiple of a solution)

\( \sin\left(\frac{n\pi x}{L}\right) \) is a solution, too.
Note: The numbers \( \lambda_n = \left( \frac{n \pi}{L} \right)^2 \) are called "eigenvalues" and the functions \( X_n(x) = \sin \left( \frac{n \pi x}{L} \right) \) are called "eigenfunctions". The reason for this follows from the ODE

\[
-\frac{d^2}{dx^2}(X) = \lambda X, \quad X(0) = X(L) = 0
\]

where \(-\frac{d}{dx^2}\) is a linear operator acting on functions. Under the linear action if a function goes to a \( \lambda \) multiple of itself, that function is called an eigenfunction (like eigenvector) with eigenvalue \( \lambda \).

We solved the ODE for \( X \). Now, let's solve ODE for \( T \).

For \( \lambda_n = \left( \frac{n \pi}{L} \right)^2 \), we have \( X_n(x) = \sin \left( \frac{n \pi x}{L} \right) \)

and \( T_n \) satisfies \( T_n' + \lambda_n \cdot k \cdot T_n = 0 \) i.e. \( T_n' = -\lambda_n \cdot k \cdot T_n \)

which gives \( T_n(t) = e^{-k \left( \frac{n \pi}{L} \right)^2 t} \)

Combining \( X_n(x) \) & \( T_n(t) \) for each \( \lambda_n \) in the product form

\[
U_n(x,t) = X_n(x) \cdot T_n(t) = \sin \left( \frac{n \pi x}{L} \right) \cdot e^{-k \left( \frac{n \pi}{L} \right)^2 t}
\]

\[
= e^{-k \left( \frac{n \pi}{L} \right)^2 t} \sin \left( \frac{n \pi x}{L} \right) \quad n=1,2,3,\ldots
\]

Our problem is linear and homogeneous, by superposition principle

\[
U(x,t) = \sum_{n=1}^{\infty} c_n U_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-k \left( \frac{n \pi}{L} \right)^2 t} \sin \left( \frac{n \pi x}{L} \right)
\]

is a general solution to the heat eqn. with given boundary conditions provided that the infinite series has the proper convergence and derivative.

We haven't finished, yet. We need to check the initial condition

**IC**

\[
u(x,0) = f(x), \quad 0 \leq x \leq L
\]
We must have

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) = f(x) \quad 0 < x < L$$

Thus, \( f(x) \) has an expansion of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right)$$

So, we must answer a few questions here.

1) Which functions can be expanded (or represented) as a sum of sine functions (maybe cosine)?

2) How can we compute coefficients?

3) What can we say about the convergence of this type of series?

All of you know the answer to (1) & (2) from MATH 254/219. (Fourier Series)

Before we study this type of series, let's see another example of separation of variables after which we'll definitely need to repeat certain things about Fourier Series and learn some more details about their convergence.

Ex: \( u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < L, \ t > 0 \)

\( u(0, t) = u(L, t) = 0 \quad t > 0 \quad \text{BC} \)

\( u(x, 0) = f(x), \ u_t(x, 0) = g(x) \quad 0 < x < L \quad \text{IC} \)

(According to BC, it's again a Dirichlet problem.)

Assume \( u(x, t) = X(x) T(t) \), then we get

$$T'' T - c^2 X'' = 0 \iff \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

1) \( X'' + \lambda X = 0 \)

2) \( T'' + \lambda c^2 T = 0 \)

Note: We keep \( c^2 \) with \( T \) since we want to have \( X \) as simple as possible so that BC \( u(0, t) = u(L, t) = 0 \) can easily be applied.
Let's start with boundary conditions

\[ u(0,t) = X(0) \cdot T(t) = 0 \iff X(0) = 0 \text{ or } T(t) = 0 \]
\[ u(L,t) = X(L) \cdot T(t) = 0 \iff X(L) = 0 \text{ or } T(t) = 0 \]

If \( T(t) = 0 \), we get the trivial solution \( U(x,t) = 0 \).
Otherwise we have a Boundary Value Problem (BVP) for \( X \).

\[ X'' + \lambda X = 0, \quad X(0) = X(L) = 0 \]

But, this is the same problem we solved for the heat eqn and solutions (or eigenfunctions) are

\[ X_n(x) = \sin\left( \frac{n\pi x}{L} \right), \quad n = 1, 2, 3, \ldots \text{ with eigenvalue } \lambda_n = \left( \frac{n\pi}{L} \right)^2 \]

Note that any multiple of \( X_n \) is a solution, too.

Now we find \( T_n(t) \) corresponding to eigenvalue \( \lambda_n = \left( \frac{n\pi}{L} \right)^2 \)

\[ T_n'' + c^2 \lambda_n T_n = 0 \iff T_n(t) = a_n \cos\left( \frac{n\pi c t}{L} \right) + b_n \sin\left( \frac{n\pi c t}{L} \right) \]

Hence, we have a sequence of solutions

\[ U_n(x,t) = X_n(x) \cdot T_n(t) = \sin\left( \frac{n\pi x}{L} \right) \cdot (a_n \cos\left( \frac{n\pi c t}{L} \right) + b_n \sin\left( \frac{n\pi c t}{L} \right)) \]

and by superposition principle we have the general solution

\[ U(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos\left( \frac{n\pi c t}{L} \right) + b_n \sin\left( \frac{n\pi c t}{L} \right) \right) \cdot \sin\left( \frac{n\pi x}{L} \right) \]

(I just named \( C_n \cdot a_n \) as \( a_n \) and \( C_n \cdot b_n \) as \( b_n \) without introducing new names for coefficients which were arbitrary)

We come to the initial conditions

\[ U(x,0) = \sum_{n=1}^{\infty} a_n \cdot \sin\left( \frac{n\pi x}{L} \right) = f(x) \]
\[ U_t(x,0) = \sum_{n=1}^{\infty} \left( -a_n \cdot \frac{n\pi c}{L} \cdot \sin\left( \frac{n\pi c t}{L} \right) + b_n \cdot \frac{n\pi c}{L} \cdot \cos\left( \frac{n\pi c t}{L} \right) \right) \cdot \sin\left( \frac{n\pi x}{L} \right) \]
Here we assume the existence of term by term derivative for the infinite series which is actually true for Fourier series but we'll prove it later.

\[ u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin \left( \frac{n\pi x}{L} \right) = g(x) \]

Hence, we come up with the same questions and we have to answer them to find the solution to both problems and more.

**FOURIER SERIES**

We'll briefly learn necessary parts of theory of Fourier series. Basically, we'll repeat the computational stuff you learned in MATH 254/219 and learn more about their "convergence." No actual proof will be given.

**Defn:** An infinite series of the form

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right)) \]

is called a Fourier series. On the set of points where the series converges, it defines a function \( f \) whose value at \( x \) is equal to the sum of the series. In this case, we say the series is the Fourier series of \( f \) and write

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right)) \]

From this equality, we can partially answer the first question to determine which functions can be represented as a Fourier series. The family of functions

\[ \left\{ \cos \left( \frac{n\pi x}{L} \right), \sin \left( \frac{n\pi x}{L} \right) \right\}_{n=1}^{\infty} \]

contains periodic functions with different periods but all have a common period \( 2L \)...
which is called “the fundamental period” of the series. Hence, by using equality \( f \) must be a periodic function with period \( 2L \) i.e. \( f(x+2L) = f(x) \).

The second property of this family is that each distinct pair of functions is orthogonal with respect to inner product

\[
\langle f, g \rangle = \int_{-L}^{L} f(x)g(x) \, dx
\]

i.e

\[
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 
0 & \text{if } m \neq n \\
L & \text{if } m = n
\end{cases}
\]

\[
\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0 
\text{ for all } m, n
\]

\[
\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 
0 & \text{if } m \neq n \\
L & \text{if } m = n
\end{cases}
\]

As a result

\[
\langle f(x), \cos\left(\frac{m\pi x}{L}\right) \rangle = \langle \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \rangle
\]

\[
= \frac{a_0}{2} \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx
\]

Keeping \( n \) fixed:

\[
\langle f(x), 1 \rangle = \frac{a_0}{2} \int_{-L}^{L} 1 \, dx = \frac{a_0}{2} \cdot 2L = a_0 \cdot L
\]

\[
\langle f(x), \sin\left(\frac{n\pi x}{L}\right) \rangle = b_n \cdot L
\]

Similarly,

\[
\langle f(x), \sin\left(\frac{m\pi x}{L}\right) \rangle = b_m \cdot L
\]
These computations show us how to compute coefficients:

\[ a_n = \frac{1}{L} \left< f(x), \cos \left( \frac{n \pi x}{L} \right) \right> = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx \]

\[ b_n = \frac{1}{L} \left< f(x), \sin \left( \frac{n \pi x}{L} \right) \right> = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx \]

Note that all formulas depend on only the interval \(-L \leq x \leq L\). Since \( f(x) \) and its Fourier series are 2L-periodic, \( f(x) \) and its Fourier series are determined for all \( x \) by its values in the interval \(-L \leq x \leq L\).

**Ex:** Assume \( f(x) = x^2 \), \(-1 < x \leq 1\). If \( f(x+2) = f(x) \) has a Fourier series. Let's find \( a_n \) & \( b_n \).

\[ a_0 = \frac{1}{1} \int_{-1}^{1} x^2 \, dx = 0 \]

\[ a_n = \begin{cases} \int_{-1}^{1} x \cos \left( \frac{n \pi x}{L} \right) \, dx & \text{even} \\ \int_{-1}^{-1} x \cos \left( \frac{n \pi x}{L} \right) \, dx & \text{odd} \end{cases} = 0 \]

\[ b_n = \begin{cases} \int_{-1}^{1} x \sin \left( \frac{n \pi x}{L} \right) \, dx & \text{odd} \\ \int_{-1}^{1} x \sin \left( \frac{n \pi x}{L} \right) \, dx & \text{even} \end{cases} = \begin{cases} 0 & \text{odd} \\ \int_{-1}^{1} x \sin \left( \frac{n \pi x}{L} \right) \, dx & \text{even} \end{cases} \\
\]

\[ b_n = 2 \left( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n \pi} \sin \left( \frac{n \pi x}{L} \right) \right) \]

We get:

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n \pi} \sin \left( \frac{n \pi x}{L} \right) \]
From this example we observe the following:

1) \( a_n = 0 \) \( n > 0 \) and this happens since \( f(x) = x \) is an odd function. Hence we can conclude that
   - If \( f(x) \) is odd on \([-L, L]\), then \( a_n = 0 \), \( n > 0 \), so just a sine series.
   - If \( f(x) \) is even on \([-L, L]\), \( b_n = 0 \) \( n > 1 \), so just a cosine series.

2) For \( x = 1 \), \( f(1) = 1 \), but \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi \cdot 1) = 0 \). Hence, Fourier series converges to a different value. How about other points? At this point we'll state a powerful theorem which will answer our questions.

**Fourier Convergence Thm:** Suppose \( f \) & \( f' \) are piecewise continuous (only finitely many jump discontinuities exist) on \( -L < x < L \) and \( f(x + 2L) = f(x) \) (periodic with period \( 2L \)). Then, \( f \) has a Fourier series whose coefficients are computed by the previously given formulas. Furthermore, the Fourier series converges to \( f(x) \) at all pts where \( f \) is continuous and it converges to

\[
\frac{f(x^+) + f(x^-)}{2} \quad (= \text{The mean value of the right and left- hand limits at the point } x)
\]

Hence \( \frac{f(1^+) + f(1^-)}{2} = \frac{(-1) + (1)}{2} = 0 \), which explains why Fourier series gives 0 above at \( x = 1 \).

Actually, this theorem gave the answers to the questions we asked before: For a piecewise cont. \( f \) which is periodic we can find a Fourier series, and we know how to compute the coefficients. Furthermore, we know the pointwise convergence of Fourier series i.e. it converges to \( f(x) \) at cont. pts.
3) \( f'(x) = 1 \) on \(-1 < x < 1\) and \( f'(x+2) = f'(x) \)

Now, if we take the derivative of the Fourier series of \( f(x) \) we get
\[
\sum_{n=1}^{\infty} (-1)^{n+1} 2 \cos(n\pi x) \]
which is clearly divergent everywhere by the \( n \)th term test. Furthermore, \( f'(x) \) has the Fourier series as itself i.e. \( a_0 = 2, a_n = 0 \) \( n \gg 1 \), \( b_n = 0 \) (\( f'(x) \) is an even function)
\[
\text{i.e. } f'(x) = 1 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x), \quad a_n = b_n = 0, n \gg 1
\]

So, we can ask: when is the derivative of the Fourier series of \( f \) equal to the derivative of \( f \)?

We know from the analysis courses if a series converges uniformly then we can do term-by-term of the series and it's equal to the derivative of the function. Here we have the following result.

**Thm:** Let \( f \in C^2([-L,L]) \) (twice continuously differentiable i.e \( f, f', f'' \) are cont. on \(-L < x < L\)) be a \( 2L \)-periodic function with \( f(L) = f(-L), f'(L) = f'(-L) \). Then the Fourier series of \( f(x) \) converges to \( f(x) \) uniformly. (Hence, we can do term-by-term derivative and also integration.)

**Ex:** Let \( f(x) = \cos^2 x \) on \([-\pi, \pi]\). Let's find its Fourier series which is of the form \( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \) (\( L = \pi \))

We know this expansion is unique i.e \( a_n \) & \( b_n \) are uniquely determined and also \( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} \)

\[
\text{ Fourier series, with } L = \pi.
\]

So, by taking \( a_0 = 1, a_1 = 0, a_2 = \frac{1}{2}, a_n = 0 \) \( n \gg 3 \)
\[
b_n = 0 \quad n \gg 1
\]

we can get its Fourier series...
Furthermore, \( \cos^2(\pi) = \cos^2(-\pi) \), and \( -2 \cos x \sin x_{x=\pi} = -2 \cos x \sin x_{x=-\pi} \)
which implies that the Fourier series converge uniformly.

Now, let's take \( f(x) = \cos^2 x \) on \([-1, 1]\) with \( f(x+2) = f(x) \).
Then, the Fourier series will be of the form

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x) \\
L = 1
\]

So, writing \( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} \) won't give the Fourier series since frequencies (multiples of \( x \) in cosine & sine) don't match.
Hence, we have to compute those ugly integrals. But, we know \( b_n = 0 \) since \( f(x) = \cos^2 x \) is even. So, we'll compute \( a_0, a_n \ n > 1 \)

\[
a_0 = \frac{1}{1} \int_{-1}^{1} \cos^2 x \, dx = \int_{-1}^{1} \left( \frac{1}{2} + \frac{\cos(2x)}{2} \right) \, dx = \frac{1}{2} x + \frac{\sin(2x)}{4} \bigg|_{-1}^{1} = \frac{1}{2} + \frac{1}{2} \sin(2)
\]

\[
a_n = \frac{1}{1} \int_{-1}^{1} \cos^2 x \cos(n\pi x) \, dx = \int_{-1}^{1} \left( \frac{1}{2} + \frac{\cos(2x)}{2} \right) \cos(n\pi x) \, dx = \int_{-1}^{1} \frac{\cos(n\pi x) + \cos(2x) \cos(n\pi x)}{2} \, dx
\]

\[
= \int_{-1}^{1} \frac{\cos(n\pi x)}{2} + \frac{1}{2} \int_{-1}^{1} \frac{\cos(2x + n\pi x) + \cos(n\pi x - 2x)}{2} \, dx
\]

\[
= 2 \cdot \sin(n\pi x) \bigg|_{0}^{1} + 2 \cdot \frac{1}{4} \left( \frac{\sin((n\pi + 2)x)}{n\pi + 2} + \frac{\sin((n\pi - 2)x)}{n\pi - 2} \right) \bigg|_{0}^{1}
\]

\[
= \frac{1}{2} \frac{\sin(n\pi + 2)}{n\pi + 2} + \frac{1}{2} \frac{\sin(n\pi - 2)}{n\pi - 2}
\]

\[
\cos^2 x = \frac{1}{2} \frac{\sin(2)}{2} + \sum_{n=1}^{\infty} \left( \frac{1}{2} \frac{\sin(n\pi + 2)}{n\pi + 2} + \frac{1}{2} \frac{\sin(n\pi - 2)}{n\pi - 2} \right) \cos(n\pi x)
\]
If \( f \) is not a periodic function, we may still write the Fourier series of \( f \).

Let \( f \) be a piecewise cont. function on \([0, L]\). Then, we can extend it to \([-L, +L]\) by defining it on \([-L, 0]\) and then make it periodic by defining it on all \( \mathbb{R} \) via \( f(x+2L) = f(x) \). But, depending on the definition over \([-L, 0]\) we'll get different Fourier series all of which converge to \( f(x) \) over \([0, L]\). If we want special Fourier series i.e. cosine or sine series, our extension can't be arbitrary.

Suppose \( f \) is given on \([0, L]\) and we want sine series of \( f \). Then, our extension must be odd to get \( a_n = 0 \) \( n > 0 \).

\[
f_{\text{ext}} = \begin{cases} 
f(x) & x \in [0, L] \\
f(-x) & x \in [-L, 0) 
\end{cases}
\]

and then we define it all over \( \mathbb{R} \) by making it \( 2L \)-periodic.

Then, Fourier series of \( f_{\text{ext}}(x) \) will be

\[
b_n = \frac{1}{L} \int_{-L}^{L} f_{\text{ext}}(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \begin{cases} 
2L & \text{odd} \\
0 & \text{even}
\end{cases} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\]

Hence, \( b_n \)'s will just depend on \( f(x) \) on \([0, L]\).

If we want cosine series of \( f \), then our extension must be even to get \( b_n = 0 \) \( n > 1 \).

\[
f_{\text{ext}}(x) = \begin{cases} 
f(x) & x \in [0, L] \\
f(-x) & x \in [-L, 0)
\end{cases}
\]

with \( f_{\text{ext}}(x+2L) = f_{\text{ext}}(x) \) to define all over \( \mathbb{R} \) with period \( 2L \).
The Fourier series of \( f(x) \) will be
\[
A_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \, dx, \quad A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx
\]

Again, \( A_n \)'s will depend on \( f(x) \) on \([-L,L]\).

Ex: \( f(x) = 1 \) on \([0, \pi]\). Let's write both sine and cosine series.

i) \( f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{L} \right) = \sum_{n=1}^{\infty} b_n \sin (nx) \)

\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin (nx) \, dx = \frac{2}{\pi} \left[ \frac{- \cos (nx)}{n} \right]_{0}^{\pi} = \frac{2}{n \pi} \left( \cos (n \pi) - \cos (0) \right) = \frac{2}{n \pi} \left( 1 - (-1)^n \right)
\]

\[
b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n \pi} & \text{if } n \text{ odd} \end{cases}
\]

\( f(x) = 1 = \sum_{k=1}^{\infty} \frac{4}{(2k-1) \pi} \sin \left( \frac{(2k-1) \pi x}{L} \right) \)

ii) \( f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n \pi x}{L} \right) \)

\[
1 = \frac{A_0}{2} + A_1 \cos (x) + A_2 \cos (2x) + \ldots
\]

So, by uniqueness of coefficients \( A_0 = 2, A_1 = A_2 = \ldots = A_n = 0 \)

Now, let's solve a wave equation since we know how to find the coefficients.

Ex: \( u_{tt} - 4u_{xx} = 0, \quad 0 < x < \pi \quad (L=\pi) \quad (c = 2) \)

BC \( u(0,t) = u(\pi,t) = 0 \quad t \geq 0 \)

IC \( u(x,0) = \sin (3x) \quad u_t(x,0) = 1 \quad 0 < x < L \)

We've found the formal solution of this Dirichlet problem

\[
u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n \pi c t}{L} \right) + b_n \sin \left( \frac{n \pi c t}{L} \right) \right) \sin \left( \frac{n \pi x}{L} \right) = \sum_{n=1}^{\infty} \left( A_n \cos (2nt) + b_n \sin (2nt) \right) \sin (nx)
\]
\[ U(x,0) = \sin(3x) = \sum_{n=1}^{\infty} a_n \sin(nx) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \cdots \]

Hence \( a_n = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases} \)

\[ U_t(x,0) = 1 = \sum_{n=1}^{\infty} \left( a_n \cdot 2n \sin(2nt) + b_n \cdot 2n \cdot \cos(2nt) \right) \sin(nx) \]

\[ = \sum_{n=1}^{\infty} b_n \cdot 2n \sin(nx) \]

\[ b_n \cdot 2n = \frac{2}{\pi} \int_0^\pi \sin(nx) \, dx \Rightarrow b_n = \frac{1}{2n} \left( \frac{2}{n\pi} \left( 1 - (-1)^n \right) \right) = \frac{1}{n^2 \pi} \left( 1 - (-1)^n \right) \]

by using previous example

\[ U(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos(2nt) + b_n \sin(2nt) \right) \sin(nx) \]

\[ = \frac{2}{\pi} \sin(2t) \sin(x) + \left( \cos(6t) \cdot \frac{2}{9\pi} \sin(6t) \right) \sin(3x) + \frac{2}{25\pi} \sin(10t) \sin(5x) \]

Ex: \[ U_t - 12U_{xx} = 0 \quad 0 < x < \pi, \quad t > 0 \]

IC \[ U(x,0) = 1 + \sin^3x \quad 0 \leq x \leq \pi \]

BC \[ U_x(0,t) = U_x(\pi,t) = 0 \quad t > 0 \]

This heat equation is different than the one we solved because of the boundary conditions (BC). This time not the temperature but the heat flow across the boundary is specified. Rigorously, the normal derivative \( \frac{\partial u}{\partial n} \), \( n \) is the outward normal, is specified on the boundary. Here \( n \) is in the direction of \( x \), so \( \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \). This type problems are called Neumann problems.
\[ U(x,0) = \sin(3x) = \sum_{n=1}^{\infty} a_n \sin(nx) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \ldots \]

Hence \( a_n = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases} \)

\[ U_t(x,0) = 1 = \sum_{n=1}^{\infty} \left( a_n \cdot 2n \sin(2nt) + b_n \cdot 2n \cos(2nt) \right) \sin(nx) \]

\[ = \sum_{n=1}^{\infty} b_n \cdot 2n \sin(nx) \]

\[ b_n \cdot 2n = \frac{2}{\pi} \int_{0}^{\pi} 1 \cdot \sin(nx) \, dx \Rightarrow b_n = \frac{1}{2n} \left( \frac{2}{n\pi} \left( 1-(-1)^n \right) \right) = \frac{1}{n^2 \pi} (1-(-1)^n) \]

\[ U(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos(2nt) + b_n \sin(2nt) \right) \sin(nx) \]

\[ = \frac{2}{\pi} \sin(2t) \sin(x) + \left( \cos(6t) + \frac{2}{9\pi} \sin(6t) \right) \sin(3x) + \frac{2}{25\pi} \sin(10t) \sin(5x) \]

**Ex:** \[ U_t - 12 U_{xx} = 0 \quad 0 < x < \pi, \ t > 0 \]

**IC** \[ U(x,0) = 1 + \cos^3 x \quad 0 \leq x \leq \pi \]

**BC** \[ U_x(0,t) = U_x(\pi,t) = 0 \quad t > 0 \]

This heat equation is different than the one we solved because of the boundary conditions (BC). This time not the temperature but the heat flow across the boundary is specified. Rigorously, the normal derivative \( \frac{\partial U}{\partial n} \), \( n \) is the outward normal, is specified on the boundary. Here \( n \) is in the direction of \( x \), so \( \frac{\partial U}{\partial n} = \frac{\partial U}{\partial x} \). This type problems are called Neumann problems.
Assume that \( u(x,t) = X(x)T(t) \). Then, we get

\[
X(x)T'(t) - 12X''(x)T(t) = 0 \iff \frac{X''}{X(x)} = \frac{T'(t)}{12T(t)} = -\lambda.
\]
This holds if and only if

\[
\frac{T'}{12T} = \frac{X''}{X} = -\lambda. \text{ Hence, we get }
\]

1) \( X'' + \lambda X = 0 \)

2) \( T' + \lambda \cdot 12T = 0 \)

Boundary conditions:

\[
U_x(0, t) = X'(0)T(t) = 0 \Rightarrow X'(0) = 0
\]

\[
U_x(\pi, t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0
\]

*If \( T(t) = 0 \), then \( u(x, t) = 0 \) i.e., we get the trivial solution which obviously doesn't satisfy the initial condition.

We again have an eigenvalue/eigenfunction problem for \( X(x) \) and we'll go case by case: \( X'' + \lambda X = 0, \ X'(0) = X'(\pi) = 0 \).

i) \( \lambda < 0 \): \( X(x) = c_1 e^{-\sqrt{-\lambda} x} + c_2 e^{\sqrt{-\lambda} x} \)

\[
0 = X(0) = -\sqrt{-\lambda} c_1 + \sqrt{-\lambda} c_2
\]

\[
0 = X'(\pi) = -\sqrt{-\lambda} c_1 e^{-\sqrt{-\lambda} \pi} + \sqrt{-\lambda} c_2 e^{\sqrt{-\lambda} \pi}
\]

has trivial solution \( c_1 = c_2 = 0 \). Hence, \( X(x) = 0 \), \( u(x, t) = 0 \).

ii) \( \lambda = 0 \): \( X(x) = c_1 + c_2 x \). \( 0 = X'(0) = c_1, \ 0 = X'(\pi) = c_2 \)

Thus, \( X(x) = c_1 \) is a nontrivial solution. (\( \lambda = 0 \) is an eigenvalue with the corresponding eigenfunction \( X_0(x) = 1 \).)

iii) \( \lambda > 0 \): \( X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \ X'(x) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda} x) + \sqrt{\lambda} c_2 \cos(\sqrt{\lambda} x) \)

\[
0 = X(0) = \sqrt{\lambda} c_2 \Rightarrow c_2 = 0, \quad 0 = X'(\pi) = -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda} \pi) \Rightarrow \sqrt{\lambda} = n \quad (c_1 = 0 \text{ gives the trivial solution.}) \quad \lambda = n^2, \ n \in \mathbb{Z}^+
\]

Hence, \( \lambda_n = n^2 \) is an eigenvalue with the corresponding eigenfunction \( X_n(x) = \cos(nx) \), for each \( n \in \mathbb{Z}^+ \).
Now, let’s find the corresponding \( T_n(t) \) for each eigenvalue \( \lambda_n \):

i) \( \lambda = 0 \)  
\[ T_0' + 0.12 T_0 = 0 \Rightarrow T_0' = 0 \Rightarrow T_0(t) = c_0. \]

Hence,  
\[ U_0(x,t) = X_0(x) \cdot T_0(t) = 1 \cdot c_0 = c_0. \]

ii) \( \lambda_n = n^2 \)  
\[ T_n' + 12n^2 T_n = 0 \Rightarrow T_n(t) = C_n \cdot e^{-12n^2 t}. \]

Hence,  
\[ U_n(x,t) = C_n \cdot e^{-12n^2 t} \cdot \cos(nx), \quad n \in \mathbb{Z}^+. \]

Thus, we have the formal series solution:

\[ U(x,t) = U_0(x,t) + \sum_{n=1}^{\infty} U_n(x,t) = c_0 + \sum_{n=1}^{\infty} C_n \cdot e^{-12n^2 t} \cdot \cos(nx). \]

Finally, we’ll try to satisfy the initial condition:

\[ U(x,0) = \frac{1}{2} + \cos^3 x = c_0 + \sum_{n=1}^{\infty} C_n \cos(nx) \]

To find \( c_0, c_1, c_2, \ldots \) we need the cosine series of \( \frac{1}{2} + \cos^3 x \).

and \( C_n = \frac{2}{\pi} \int_0^{\pi} (1 + \cos^3 x) \cdot \cos(nx) \, dx \), but \( c_0 = \frac{1}{\pi} \int_0^{\pi} (1 + \cos^3 x) \, dx \)

(Note that \( c_0 = \frac{a_0}{2} \), and \( a_0 = \frac{2}{L} \int_0^{L} f(x) \, dx \), for cosine series.)

But, if we can write \( 1 + \cos^3 x \) in terms of \( \cos(nx) \) using some trigonometric identities, then there’s no need to compute the integrals by uniqueness of Fourier coefficients.

\[ 1 + \cos^3 x = 1 + \cos x \cdot \cos^2 x = 1 + \cos x \left( \frac{1}{2} + \frac{1}{2} \cos(2x) \right) \]

\[ = 1 + \frac{1}{2} \cos x + \frac{1}{2} \cos x \cdot \cos(2x) \]

\[ = 1 + \frac{1}{2} \cos x + \frac{1}{2} \left( \frac{1}{2} \left( \cos(2x-x) + \cos(2x+x) \right) \right) \]

\[ = 1 + \frac{3}{4} \cos x + \frac{1}{4} \cos(3x), \quad \text{so} \quad c_0 = 1 \left( = \frac{a_0}{2} \right) \]

\[ c_1 = \frac{3}{4}, \quad \text{all other } c_n = 0 \]

\[ c_3 = \frac{1}{4} \]
Hence, \( U(x, t) = 1 + \frac{3}{4} e^{-12t} \cos(x) + \frac{1}{4} e^{-123^2t} \cos(3x) \)

**Non-homogeneous Equations with Homogeneous Boundary Conditions**

Here we use a general idea due to Fourier which basically says the solution can be expanded with respect to eigenfunctions of the corresponding (homogeneous) boundary value problem. (It may look like using variation of parameters method with infinitely many functions coming from homogeneous solution.)

Ex: \( u_{tt} - c^2 u_{xx} = h(x, t), \quad 0 < x < L, \quad t > 0 \)
\( u(0, t) = u(L, t) = 0, \quad t > 0 \).
\( u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L \)

For \( h(x, t) = 0 \), the boundary value problem we solved before \( (x'' + \lambda x = 0, \quad x(0) = x(L) = 0) \) gave \( \lambda_n = \left( \frac{n \pi}{L} \right)^2 \) as eigenvalues with corresponding eigenfunctions \( X_n(x) = \sin \left( \frac{n \pi}{L} x \right) \quad n = 1, 2, 3, \ldots \)

We assume \( u(x, t) = \sum_{n=1}^{\infty} U_n(t) \cdot \sin \left( \frac{n \pi}{L} x \right) \) for some coefficient functions \( U_n(t) \) (which is somehow guaranteed in our cases by completeness theorems of Fourier analysis.) where

\[
U_n(t) = \frac{2}{L} \int_0^L u(x, t) \sin \left( \frac{n \pi}{L} x \right) \, dx.
\]

Similary, we assume

\[
h(x, t) = \sum_{n=1}^{\infty} h_n(t) \sin \left( \frac{n \pi}{L} x \right), \quad h_n(t) = \frac{2}{L} \int_0^L h(x, t) \sin \left( \frac{n \pi}{L} x \right) \, dx
\]

\[
f(x) = \sum_{n=1}^{\infty} f_n \sin \left( \frac{n \pi}{L} x \right), \quad f_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n \pi}{L} x \right) \, dx
\]

\[
g(x) = \sum_{n=1}^{\infty} g_n \sin \left( \frac{n \pi}{L} x \right), \quad g_n = \frac{2}{L} \int_0^L g(x) \sin \left( \frac{n \pi}{L} x \right) \, dx
\]
Then, we substitute them into the equation (assuming uniform convergence so we can do term by term derivative)

\[ \sum_{n=1}^{\infty} U_n''(t) \cdot \sin \left( \frac{n \pi x}{L} \right) - c^2 \sum_{n=1}^{\infty} U_n(t) \cdot \left( \frac{n \pi }{L} \right)^2 \sin \left( \frac{n \pi x}{L} \right) = \sum_{n=1}^{\infty} h_n(t) \sin \left( \frac{n \pi x}{L} \right) \]

with

\[ U(x,0) = \sum_{n=1}^{\infty} U_n(0) \cdot \sin \left( \frac{n \pi x}{L} \right) = f(x) = \sum_{n=1}^{\infty} f_n \sin \left( \frac{n \pi x}{L} \right) \]

\[ U_t(x,0) = \sum_{n=1}^{\infty} U_n'(0) \cdot \sin \left( \frac{n \pi x}{L} \right) = g(x) = \sum_{n=1}^{\infty} g_n \sin \left( \frac{n \pi x}{L} \right) \]

Comparing coefficients of the eigenfunction \( \sin \left( \frac{n \pi x}{L} \right) \) we get

(*) \[ U_n'' + c^2 \left( \frac{n \pi }{L} \right)^2 U_n = h_n(t) \quad \text{with} \quad U_n(0) = f_n, \quad U_n'(0) = g_n \]

Note that \( h_n(t), f_n \) and \( g_n \) are already known when \( h, f, g \) are given.

Solving (*), \[ U_n(t) = c_1 \cos \left( \frac{n \pi c t}{L} \right) + c_2 \sin \left( \frac{n \pi c t}{L} \right) + U_n^p(t) \]

where \( U_n^p(t) \) is the particular solution. with \( U_n(0) = f_n, \quad U_n'(0) = g_n \)

\[ U(x,t) = \sum_{n=1}^{\infty} U_n(t) \cdot \sin \left( \frac{n \pi x}{L} \right) \]

**Example:**

\[ U_{tt} - U_{xx} = \sin \left( 2 \pi x \right) \cdot \sin \left( 2t \right), \quad 0 < x < 1, \quad t > 0 \]

\[ U(0,t) = U(1,t) = 0 \quad t > 0 \]

\[ U(x,0) = 0, \quad U_t(x,0) = 0 \]

Soln: Here eigenvalues are \( (n\pi)^2 \) and eigenfunctions are \( X_n(x) = \sin(n\pi x) \)

for the boundary value problem. Hence, assume

\[ U(x,t) = \sum_{n=1}^{\infty} U_n(t) \cdot \sin \left( n \pi x \right), \]

\[ h(x,t) = \sum_{n=1}^{\infty} h_n(t) \cdot \sin \left( n \pi x \right) = \sin \left( 2t \right) \cdot \sin \left( 2 \pi x \right) \]

(Clearly, \( h_2(t) = \sin \left( 2t \right) \) and \( h_n(t) = 0 \quad n \neq 2 \) )
\[ 0 = \hat{f}(x) = \sum_{n=1}^{\infty} \hat{f}_n \sin(n \pi x) \Rightarrow \hat{f}_n = 0 = U_n(0) \]
\[ 0 = \hat{g}(x) = \sum_{n=1}^{\infty} \hat{g}_n \sin(n \pi x) \Rightarrow \hat{g}_n = 0 = U_n'(0) \]

Now if we substitute we get:
\[ \sum_{n=1}^{\infty} \left( U_n'' + (n \pi)^2 U_n \right) \sin(n \pi x) = \sin(2t) \cdot \sin(2\pi x) \]

\[ n \neq 2 \quad U_n'' + (n \pi)^2 U_n = 0, \quad U_n(0) = U_n'(0) = 0 \]

\[ U_n(t) = c_1 \cos(n \pi t) + c_2 \sin(n \pi t) \]. Plugging the initial conditions we get \( c_1 = c_2 = 0 \)

\[ n = 2 \quad U_2'' + (2 \pi)^2 U_2 = \sin(2t), \quad U_2(0) = U_2'(0) = 0 \]

\[ U_2(t) = c_1 \cos(2 \pi t) + c_2 \sin(2 \pi t) + U_2'(t) \]

We can find \( U_2'(t) \) by using the method of undetermined coeff. or by using variation of parameters.

Let's use the former. \( U_2'(t) = A \sin(2t) + B \cos(2t) \)

Plug in:
\[ -4A \sin(2t) - 4B \cos(2t) + 4\pi^2 A \sin(2t) + 4\pi^2 B \cos(2t) = \sin(2t) \]

\[ B = 0, \quad A = \frac{1}{4\pi^2 - 4} \]

\[ U_2(t) = c_1 \cos(2 \pi t) + c_2 \sin(2 \pi t) + \frac{1}{4\pi^2 - 4} \sin(2t) \]

\[ 0 = U_2(0) = c_1 \]
\[ 0 = U_2'(0) = 2\pi c_2 + \frac{2}{4\pi^2 - 4} \]

\[ \therefore U(x, t) = \frac{1}{4\pi^2 - 4} \left( -\frac{1}{\pi} \sin(2\pi t) + \sin(2t) \right) \cdot \sin(2\pi x) \]
Ex: \( U_t - U_{xx} = t \cdot \cos(3x) \quad 0 < x < \pi, \ t > 0 \)

\[ U(x,0) = 1 \quad , 0 < x < \pi \]

\[ U_x(0,t) = U_x(\pi,t) = 0 \]

Soln: If we apply separation of variables \( U(x,t) = X(x)T(t) \) boundary value problem will be \( X'' + \lambda X = 0, \ X'(0) = X'(\pi) = 0 \)
This problem has eigenvalues \( \lambda_n = n^2 \), eigenfunctions \( X_n = \cos(nx) \) \( n = 0, 1, 2, \ldots \) (Look page 78) Note \( n = 0, \ X_0 = 1 \).

Assume \( U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t) \cdot \cos(nx) \)

\[ t \cdot \cos(3x) = h(x,t) = \frac{h_0(t)}{2} + \sum_{n=1}^{\infty} h_n(t) \cdot \cos(nx) \]

\[ = \frac{h_0(t)}{2} + h_1(t) \cdot \cos x + h_2(t) \cdot \cos(2x) + h_3(t) \cdot \cos(3x) + \ldots \]

So, \( h_3(t) = t \), and \( h_n(t) = 0, \ n \neq 3 \).

\[ U(x,0) = f(x) = 1 = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cdot \cos(nx) \Rightarrow f_n = 0 \quad n \neq 0 \]

\[ f_0 = 2 \quad n = 0 \]

Let's plug into the equation

\[ \sum_{n=1}^{\infty} U_n'(t) \cdot \cos(nx) - \sum_{n=1}^{\infty} U_n(t) \cdot (-n^2) \cdot \cos(nx) = t \cdot \cos(3x) \]

\[ \sum_{n=1}^{\infty} (U_n' + n^2 U_n) \cdot \cos(nx) = t \cdot \cos(3x) \]

\[ U_n' + n^2 U_n = \begin{cases} t & n = 3 \\ 0 & n \neq 3 \end{cases} \]

\[ n \neq 3: \ U_n' + n^2 U_n = 0 \Rightarrow U_n(t) = C_n e^{-n^2 t} \quad \Rightarrow \ U_n(0) = C_n \]

\[ \begin{bmatrix} U(x,0) = 1 = \frac{U_0(0)}{2} + \sum_{n=1}^{\infty} U_n(0) \cdot \cos(nx) \Rightarrow U_0(0) = 2 = f_0 \\ U_n(0) = 0 = f_n \quad n \neq 0 \end{bmatrix} \]
\[ n=0 \quad U_0(t)=2=C_0 \quad \Rightarrow \quad U_0(t)=2e^{-\frac{at}{2}} = 2. \]

\[ n \neq 0,3 \quad U_n(0)=0=C_n \quad \Rightarrow \quad U_n(t)=0 \quad n \neq 3 \]

\[ n=3 \quad U_3(t)+9U_3=t, \quad U_3(0)=0 \]

\[ U_3(t)=c_3e^{-\frac{at}{2}}+U_3^p(t). \]

We can find \( U_3^p(t) \) by the method of undetermined coefficients:

\[ U_3^p(t)=At+B \]

Plug in: \( A+9.(At+B)=t \quad \Rightarrow \quad A=\frac{1}{9} \quad B=-\frac{1}{81} \)

So, \( U_3(t)=c_3e^{-\frac{at}{2}}+\frac{1}{9}t-\frac{1}{81} \)

\[ 0=U_3(0)=c_3-\frac{1}{81} \quad \Rightarrow \quad c_3=\frac{1}{81} \quad \therefore \quad U_3(t)=\frac{1}{81}e^{-\frac{at}{2}}+\frac{1}{9}t-\frac{1}{81} \]

Hence,

\[ U(x,t)=\frac{U_0(t)}{2}+\sum_{n=1}^{\infty} U_n(t) \cos(nx) = \frac{2}{2} + \left( \frac{1}{81}e^{-\frac{at}{2}}+\frac{1}{9}t-\frac{1}{81} \right) \cos(3x) + 0.0. \]

Note: As you can see from examples, only certain problems can be solved with this method. Actually, there's more, but those can be done by using Green's identities. I'll give an example of it later.

Exercise: \( U_{tt}-4 U_{xx}=x.t, \quad 0<x<1, \quad t>0 \)

\[ U(0,t)=U(1,t)=0 \quad t>0 \]

\[ U(x,0)=U_t(x,0)=0 \quad 0\leq x<1 \]

(1's a little bit harder and longer than the examples above.)
Non-homogeneous Boundary Conditions

Method: We introduce a new dependent variable \( V(x,t) = U(x,t) - w(x,t) \)
\( w(x,t) \) is chosen in such a way that \( V(x,t) \) satisfies homogeneous boundary conditions. This can be done by looking for an auxiliary simple function \( w(x,t) \) satisfying only the given non-homogeneous boundary conditions. In fact, we can always find such a function \( w(x,t) \) that has the form
\[
 w(x,t) = (A_1 + B_1 x + C_1 x^2) a(t) + (A_2 + B_2 x + C_2 x^2) b(t)
\]
where \( a(t) \& b(t) \) are the functions which are defining the boundary conditions.

Example: Dirichlet: \( U(0,t) = a(t) \quad U(L,t) = b(t) \)
We want \( w(0,t) = a(t) = A_1 a(t) + A_2 b(t) \Rightarrow A_2 = 0, A_1 = 1 \)
\( w(L,t) = b(t) = (1 + B_1 L + C_1 L^2) a(t) + (B_2 L + C_2 L^2) b(t) \)
can be achieved by \( B_2 = \frac{1}{L}, C_2 = C_1 = 0, B_1 = -\frac{1}{L} \)
So, we get \( w(x,t) = a(t) + \frac{x}{L} (b(t) - a(t)) \)

Neumann: \( U_x(0,t) = a(t), \quad U_x(L,t) = b(t) \)
\( w(x,t) = x a(t) + \frac{x^2}{2L} (b(t) - a(t)) \) satisfies the given conditions.

Robin: \( U(0,t) = a(t), \quad U_x(L,t) = b(t) \)
\( w(x,t) = a(t) + x b(t) \) satisfies the given conditions.

Ex: Consider \( u_{tt} - u_{xx} = 1 + x \cdot \cos t \quad 0 < x < 2, \quad t > 0 \)
\( u(0,t) = 5, \quad u(2,t) = t \quad t > 0 \)
\( u(x,0) = 1 + \cos (\pi x) \quad 0 \leq x \leq 2 \)

Transform it to a problem with homogeneous boundary conditions.
This is a Dirichlet type boundary value problem. So, we can choose \( w(x,t) = 5 + \frac{x}{2}(t-5) \) \( (w(0,t) = 5 \quad w(2,t) = t) \)

\[
u(x,t) = u(x,t) - \left( 5 + \frac{x}{2} (t-5) \right) \quad u(x,t) = u(x,t) + \left( 5 + \frac{x}{2} (t-5) \right)
\]

Now plug in:

\[
\left( u(x,t) + \left( 5 + \frac{x}{2} (t-5) \right) \right)_{t} - \left( u(x,t) + \left( 5 + \frac{x}{2} (t-5) \right) \right)_{xx} = 1 + x \cdot \cos t
\]

\[
u_{t} - \nu_{xx} = 1 + x \cdot \cos t - \frac{x}{2} \quad \text{(Eqn.)}
\]

\( u(0,t) = u(0,t) + 5, t = u(2,t) = u(2,t) + t \)

\[
u(0,t) = v(2,t) = 0 \quad \text{(BC)}
\]

\[
1 + \cos(tx) = u(x,0) = v(x,0) + \left( 5 + \frac{x}{2} (0-5) \right)
\]

\[
v(x,0) = 1 + \cos(tx) - \left( 5 - \frac{5}{2} x \right) \quad \text{(IC)}
\]

**Special Case:** \( a(t) = a \quad b(t) = b \) i.e constant functions

\[ h(x,t) = h(x) \quad \text{(external force/source is a function of } x) \]

In this case, \( w(x,t) \) can be chosen in such a way that

\[ w(x,t) = w(x) \] and both the eqn and the boundary conditions become homogeneous.

**Example:**

\[
u_{tt} - \nu_{xx} = x \quad 0 < x < 1, t > 0
\]

\[
u(0,t) = 1, \nu(1,t) = 2 \quad t > 0
\]

\[
u(x,0) = f(x) \quad \nu_{t}(x,0) = g(x)
\]

Let \( u(x,t) = u(x,t) - w(x) \), where

\[
(w(x))_{t} - (w(x))_{xx} = x \quad \Rightarrow \quad -w'' = x \quad \Rightarrow \quad w(x) = C_{1} + C_{2} x + W_{p}(x)
\]

\[
w(0) = 1 \quad w(1) = 2
\]

\[
w(0) = 1 \quad w(1) = 2
\]
By the method of undetermined coefficients \( W(x) = x^2(Ax + B) = Ax^3 + Bx^2 \)

Plug in: \(-6Ax + 2B = x \) \( \Rightarrow A = -\frac{1}{6} \) \( B = 0 \)

\[ W(x) = C_1 + C_2x - \frac{1}{6}x^3 \]

\[ 1 = w(0) = C_1 \]
\[ 2 = w(1) = C_1 + C_2 - \frac{1}{6} \]

\( \therefore C_1 = 1, C_2 = \frac{7}{6} \)

\[ U(x,t) = W(x,t) + \left( 1 + \frac{7}{6}x - \frac{1}{6}x^3 \right) \]

\[ U_{tt} - U_{xx} = U_{tt} - U_{xx} = 0 \]

\[ 1 = U(0,t) = W(0,t) + 1 \Rightarrow W(0,t) = 0 \]

\[ 2 = U(1,t) = W(1,t) + 2 \Rightarrow W(1,t) = 0 \]

\[ f(x) = u(x,0) = W(x,0) + \left( 1 + \frac{7}{6}x - \frac{1}{6}x^3 \right) \Rightarrow W(x,0) = f(x) - \left( 1 + \frac{7}{6}x - \frac{1}{6}x^3 \right) \]

\[ g(x) = U_t(x,0) = W_t(x,0) + 0 \Rightarrow W_t(x,0) = g(x) \]

\[ \textbf{Ex.} \text{ Solve } U_t - 4U_{xx} = 10 \text{, } 0 \leq x \leq 2, \ t > 0 \]

\[ U(x,0) = \begin{cases} 0 & 0 \leq x \leq 2 \\ U(0,t) = 0, \ U(1,t) = 1 & t > 0 \end{cases} \]

Let \( U(x,t) = W(x,t) + W(x) \) and \(-4W'' = 10\), \( W(0) = 0\), \( W(2) = 1\)

So, \( W(x) = C_1 + C_2x + W(x) \). By the method of und. coeff. \( W(x) = x^2 A \)

Plug in: \(-4 \cdot 2A = 10 \Rightarrow A = -\frac{10}{8} = -\frac{5}{4} \)

\[ W(x) = C_1 + C_2x - \frac{5}{4}x^2 \]

\[ 0 = W(0) = C_1 \]
\[ 1 = W(2) = C_1 + 2C_2 - 5 \frac{5}{4} \]

\( \therefore C_1 = 0, C_2 = \frac{3}{2} \)

\[ \therefore W(x) = 3x - \frac{5}{4}x^2 \]
\[ U(x, t) = u(x, t) + 3x - \frac{5}{4}x^2 \]

\[ U_t - 4U_{xx} = U_t - 4\left(\frac{U_{xx}}{4}\right) = 10 \Rightarrow U_t - 4U_{xx} = 0 \]

\[ U(0, t) = 0 = U(0, t) + 0 \Rightarrow U(0, t) = 0 \]

\[ U(2, t) = 1 = U(2, t) + 6 - 5 \Rightarrow U(2, t) = 0 \]

\[-\frac{5}{4}x^2 = U(x, 0) = u(x, 0) + 3x - \frac{5}{4}x^2 \Rightarrow U(x, 0) = -3x \]

Homogeneous Heat Eqn: \[ U(x, t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{2}\right)^2 4t} \sin\left(\frac{n\pi}{2}x\right) \]

\[ L = 2 \]

\[ u = x \]

\[ dv = \sin\left(\frac{n\pi}{2}x\right) dx \]

\[ du = dx \]

\[ v = -\cos\left(\frac{n\pi}{2}x\right) \]

\[ C_n = \frac{2}{2} \left[ \int_{0}^{2}(-3x) \sin\left(\frac{n\pi}{2}x\right) dx \right] = -3 \int_{0}^{2} x \sin\left(\frac{n\pi}{2}x\right) dx \]

\[ = -3 \left( -\frac{2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right) + \left[ \frac{2}{n\pi} \right]^{2} \cos\left(\frac{n\pi}{2}x\right) dx \]

\[ = -3 \left( -\frac{2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right) \]

\[ = -3 \left( -\frac{4}{n\pi} \cos\left(n\pi\right) - 0 \right) = (-1)^n \left( \frac{12}{n\pi} \right) \]

\[ U(x, t) = \sum_{n=1}^{\infty} \left( -1 \right)^n \frac{12}{n\pi} e^{-\left(\frac{n\pi}{2}\right)^2 4t} \sin\left(\frac{n\pi}{2}x\right) + 3x - \frac{5}{4}x^2 \]