M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	MidTerm II	13.05.2017	13:30

Last Name : Name : Student No:	Signature : Section : Duration : 119 *minutes*	

5 QUESTIONS ON 4 PAGES	TOTAL 100 POINTS

1	2	3	4	5	SHOW YOUR WORK		

1. (3+4+6+6+6 pts) Given $f(x) = \dfrac{x^3-1}{x^3+1}$ with $f'(x) = \dfrac{6x^2}{(x^3+1)^2}$ and $f''(x) = \dfrac{12x(1-2x^3)}{(x^3+1)^3}$.

(a) Find the domain and intercepts of $f(x)$.

$\text{Dom}(f) = \mathbb{R}\setminus\{-1\}$, x-intercept: $y=0$, $x=1$

y-intercept: $x=0$, $y=-1$

(b) Find all asymptotes of $f(x)$.

Vertical Asymptote: $\lim_{x\to -1^+} \dfrac{x^3-1}{x^3+1} = -\infty$ $\lim_{x\to -1^-} \dfrac{x^3-1}{x^3+1} = +\infty$ i.e. $x=-1$

Horizontal Asymptote: $\lim_{x\to \mp\infty} \dfrac{x^3-1}{x^3+1} = \lim_{x\to\mp\infty} \dfrac{\cancel{x^3}(1-1/x^3)}{\cancel{x^3}(1+1/x^3)} = 1$ i.e. $y=1$

(c) Find the intervals of increase and decrease and the local extreme values of $f(x)$.

$f'(x)=0 \Rightarrow x=0$, $f'(x)$ doesn't exist when $x=-1$

		-1		0	
$f'(x)$	$+$ ↗		$+$ ↗		$+$ ↗

Always increasing on $(-\infty,-1)\cup(-1,+\infty)$

No local extremas.

(d) Find the intervals of concavity and the inflection points of $f(x)$.

$f''(x)=0 \Rightarrow x=0$, $x=\dfrac{1}{\sqrt[3]{2}}$, $f''(x)$ doesn't exist when $x=-1$

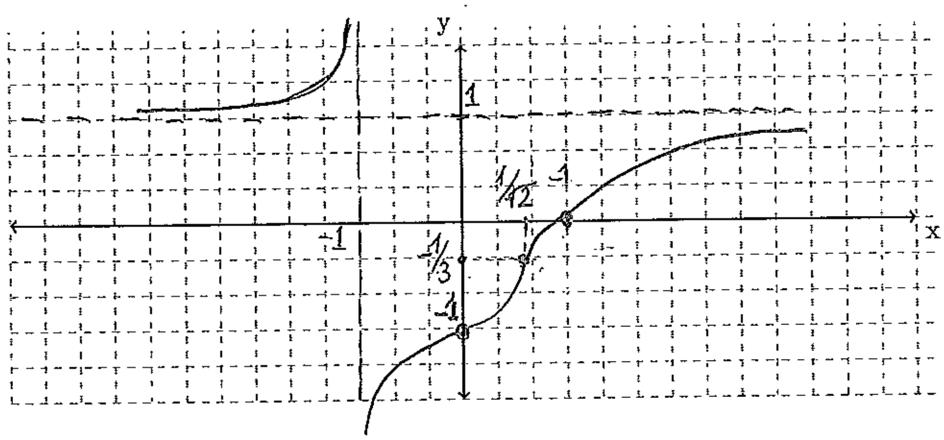
		-1		0		$1/\sqrt[3]{2}$	
f''	$+$		$-$		$+$		$-$

Concave up on $(-\infty,-1)\cup(0, 1/\sqrt[3]{2})$

Concave down on $(-1,0)\cup(1/\sqrt[3]{2}, +\infty)$

0 & $1/\sqrt[3]{2}$ are inflection pts.

(e) Sketch the graph of $f(x)$.



$f(0) = -1$

$f(1/\sqrt[3]{2}) = -1/3$

Question 2. **(6+6+7+6 pts)** Evaluate the following limits.

(a) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{\ln x\cdot(x-1)}\ \left(\frac{0}{0}\right) \overset{\text{L'Hos.}}{=} \lim_{x\to 1^+} \frac{1-1/x}{\frac{1}{x}(x-1)+\ln x}\ \left(\frac{0}{0}\right)$

$\overset{\text{L'Hos.}}{=} \lim_{x\to 1^+} \frac{1/x^2}{+1/x^2+\frac{1}{x}} \underset{\text{cont. at } 1}{=} \frac{1}{2}$

(b) $\lim_{x\to 0^+}(1-3x)^{\frac{1}{x}}\ (1^\infty) = \lim_{x\to 0^+}\left(e^{\ln(1-3x)}\right)^{1/x} = \lim_{x\to 0^+} e^{\frac{\ln(1-3x)}{x}} = e^{-3}.$

e^x is cont. at -3

$\left[\lim_{x\to 0^+}\frac{\ln(1-3x)}{x}\ \left(\frac{0}{0}\right) = \lim_{x\to 0^+}\frac{\frac{-3}{1-3x}}{1} \underset{\text{cont. at } 0}{=} -3\right]$

(c) $\lim_{x\to 0}\frac{\sqrt{1-\tan x}-\sqrt{1+\tan x}}{\sin x}\ \frac{(\sqrt{1-\tan x}+\sqrt{1+\tan x})}{(\sqrt{1-\tan x}+\sqrt{1+\tan x})} = \lim_{x\to 0}\frac{(1-\tan x)-(1+\tan x)}{\sin x\,(\sqrt{1-\tan x}+\sqrt{1+\tan x})}$

$= \lim_{x\to 0}\frac{-2\tan x}{\sin x\cdot(\sqrt{1-\tan x}+\sqrt{1+\tan x})} = \lim_{x\to 0}\frac{-2}{\cos x\cdot(\sqrt{1-\tan x}+\sqrt{1+\tan x})}$

$\underset{\text{cont. at } 0}{=} -1$

(d) $\lim_{x\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{2+\frac{3i}{n}} = \int_a^b f(x)\,dx$, where

$\Delta x = \frac{3}{n} = \frac{b-a}{n} \qquad x_i = a+i\cdot\Delta x = a+\frac{3i}{n}$

$f(x_i) = \sqrt{2+\frac{3i}{n}} = \sqrt{2+i\Delta x} = \sqrt{x_i} \Rightarrow f(x) = \sqrt{x}$ $\quad a=2,\ b=5$

$\int_2^5 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_2^5 = \frac{2}{3}\left(\sqrt{125}-\sqrt{8}\right)$

Hence, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{2+\frac{3i}{n}} = \frac{2}{3}\left(\sqrt{125}-\sqrt{8}\right)$

3.(6+6+6+7 pts) Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^2} = \int \frac{2(u-1)\,du}{u^2} = 2\int\left(\frac{1}{u}-\frac{1}{u^2}\right)du = 2\left(\ln|u|+\frac{1}{u}\right)+C$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2(u-1)\,du$

$$= 2\left(\ln(1+\sqrt{x}) + \frac{1}{1+\sqrt{x}}\right) + C$$

(b) $\displaystyle\int_0^{\ln 2} e^{2x} e^{(e^x)}\,dx = \int_1^2 t e^t\,dt = e^t \cdot t\Big|_1^2 - \int_1^2 e^t\,dt = (e^t \cdot t - e^t)\Big|_1^2$

$t = e^x \quad dt = e^x\,dx$ $\quad u = t \quad du = dt$

$dv = e^t\,dt \quad v = e^t$

$$= (2e^2 - e^2) - (e - e) = e^2.$$

(c) $\displaystyle\int \frac{\cos x + \cos^3 x}{\sin^3 x - \sin^2 x}\,dx = \int \frac{du}{u^3-u^2} + \int \frac{(1-u^2)\,du}{u^3-u^2} = \int \frac{-u^2+2}{u^2(u-1)}\,du$

$u = \sin x$

$du = \cos x\,dx$

$\dfrac{A}{u} + \dfrac{B}{u^2} + \dfrac{C}{u-1} = \dfrac{-u^2+2}{u^2(u-1)} \Rightarrow \dfrac{Au^2 + Cu^2 - Au + Bu - B}{u^2(u-1)} = \dfrac{-u^2+2}{u^2(u-1)} \Rightarrow \begin{aligned} -B &= 2 \\ -A+B &= 0 \\ A+C &= -1 \end{aligned}$

$u(u-1) \quad (u-1) \quad (u^2)$

we get $\left.\begin{aligned} B &= -2 \\ A &= -2 \\ C &= 1 \end{aligned}\right\} \Rightarrow \displaystyle\int \frac{-u^2+2}{u^2(u-1)}\,du = -2\ln|u| + 2\,\frac{1}{u} + \ln|u-1| + C = \ln\left(\frac{1}{(\sin x)^2}\right) + \frac{2}{\sin x} + \ln|\sin x - 1| + C$

(d) $\displaystyle\int \frac{x \ln x}{\sqrt{x^2-1}}\,dx = \int \frac{\ln(\sec\theta)\cdot\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta - 1}} = \int \frac{\ln(\sec\theta)\cdot\sec^2\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}}$

$x = \sec\theta$

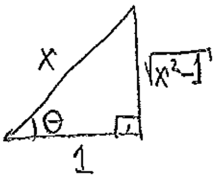
$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \ln(\sec\theta)\cdot\sec^2\theta\,d\theta = \ln(\sec\theta)\cdot\tan\theta - \int \tan^2\theta\,d\theta$$

$u = \ln(\sec\theta) \quad du = \tan\theta\,d\theta$

$dv = \sec^2\theta \quad v = \tan\theta$

$$= \ln(\sec\theta)\cdot\tan\theta - \int(\sec^2\theta - 1)\,d\theta = \ln(\sec\theta)\cdot\tan\theta - \tan\theta + \theta + C$$



$$= \ln(x)\cdot\sqrt{x^2-1} - \sqrt{x^2-1} + \sec^{-1}(x) + C$$

$\theta = \sec^{-1}(x)$

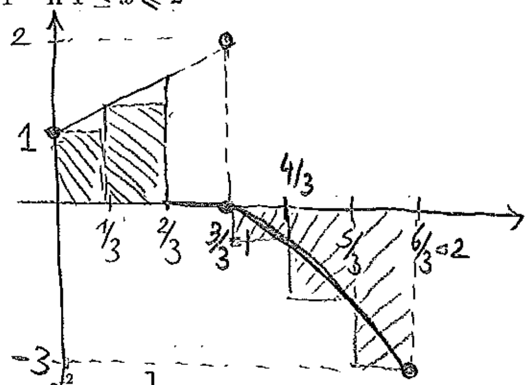
4.(6+9 pts) This problem has two **unrelated** parts.

(a) Compute the lower Riemann sum for the function $f(x) = \begin{cases} x+1 & \text{if } 0 \le x < 1 \\ -x^2+1 & \text{if } 1 \le x \leqslant 2 \end{cases}$ on [0,2] corresponding to partition P_6 of [0,2] into 6 subintervals of equal length.

$$L(f,P_6) = \frac{1}{3}\cdot\left[f(0) + f(1/3) + f(1) + f(4/3) + f(5/3) + f(2)\right]$$

$$= \frac{1}{3}\left[1 + \frac{4}{3} + 0 - \frac{7}{9} - \frac{16}{9} - 3\right]$$

$$= \frac{1}{3}\left[\frac{9+12+0-7-16-27}{9}\right] = \frac{-29}{27}$$



(b) Show that $F(x)$ has a local maximum at $x = -2$, where $F(x) = \int_0^x \left[t^2 \cdot \int_4^{t^2} e^{u^2} du\right] dt$.

$$F'(x) = x^2 \cdot \int_4^{x^2} e^{u^2} du \qquad F'(-2) = (-2)^2 \int_4^{(-2)^2=4} e^{u^2} du = 0 \qquad x=-2 \text{ is a critical pt.}$$

$$F''(x) = 2x \int_4^{x^2} e^{u^2} du + x^2 \cdot e^{x^4} \cdot 2x \qquad F''(-2) = 2\cdot(-2)\int_4^{(-2)^2} e^{u^2} du + (-2)^2 e^{(-2)^4} \cdot (2\cdot(-2))$$

$$= -16e^{16} < 0$$

Hence, by 2^{nd} Derivative Test, $x=-2$ is a local maximum pt.

5.(10 pts) There are 50 orange trees in a garden. Each tree produces 800 oranges. For each additional tree planted in the garden, the output per tree **drops** by 10 oranges. How many trees should be added to the existing garden in order to maximize the total output of trees?

$x \equiv$ number of orange trees, output per tree $= 800 - 10x$

$$T(x) = (50 + x)(800 - 10x) = 40000 + 300x - 10x^2 \qquad 0 \le x \le 80$$

$$T'(x) = 300 - 20x = 0 \Rightarrow x = 15$$

$T(x)$ is continuous on $[0,80]$ By Extereme Value Theorem, the absolute max/min exist.

$T(0) = 40000$

$T(80) = 0$

$T(15) = 42250 \rightsquigarrow$ the global maximum value

Hence, 15 more trees should be planted to maximize the total.

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Signature : ....