M E T U Department of Mathematics

Math 119	Calculus with Analytic Geometry	Final	01.06.2017	13:30

Last Name : Name : Student No:	Signature : Section : Duration : 119 *minutes*	

5 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	SHOW YOUR WORK		

1. (8+8+9 pts) This problem has three unrelated parts.

(a) If $\displaystyle\lim_{x\to 2}\frac{\int_x^2 f(t)dt}{x^2-4}=3,$ **find** $\displaystyle\lim_{x\to 2} f(x)$.

$$3=\lim_{x\to 2}\frac{\int_x^2 f(t)dt}{x^2-4}\left(\frac{0}{0}\right)\overset{\text{L'Hospital}}{=}\lim_{x\to 2}\frac{-f(x)}{2x}=\lim_{x\to 2}\frac{f(x)}{-2x}$$

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{f(x)}{-2x}\cdot(-2x)=\left[\lim_{x\to 2}\frac{f(x)}{-2x}\right]\cdot\left[\lim_{x\to 2}-2x\right]=3\cdot(-4)=-12.$$

$-2x$ is cont.

(b) Write the equation of the tangent line to the curve $f(x)=(\arctan x)^{\ln x}$ at the point $(1,1)$.

$$f(x)=\left(e^{\ln(\arctan x)}\right)^{\ln x}=e^{\ln x\cdot\ln(\arctan x)}$$

$$f'(x)=e^{\ln x\cdot\ln(\arctan x)}\cdot\left(\frac{1}{x}\cdot\ln(\arctan x)+\ln x\cdot\frac{\frac{1}{1+x^2}}{\arctan x}\right)\Rightarrow f'(1)=e^0\cdot\left(1\cdot\ln(\pi/4)-0\right)$$

Tangent Line Equation: $y-1=\ln\left(\frac{\pi}{4}\right)\cdot(x-1)$

(c) Is there any number $c\in\mathbb{R}$ which makes the function

$$f(x)=\begin{cases}\dfrac{(\sin x)^7}{\sin(x^7)} & \text{if } -\dfrac{\pi}{4}<x<0\\ c & \text{if } x=0\\ \dfrac{1-\cos x}{\ln(1+x^2)} & \text{if } 0<x<\dfrac{\pi}{4}\end{cases}$$

continuous at $x=0$? **Explain.**

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{(\sin x)^7}{\sin(x^7)}=\lim_{x\to 0^-}\frac{\frac{\sin^7 x}{x^7}}{\frac{\sin(x^7)}{x^7}}=\lim_{x\to 0^-}\frac{\left(\frac{\sin x}{x}\right)^7}{\frac{\sin(x^7)}{(x^7)}}=\frac{1^7}{1}=1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1-\cos x}{\ln(1+x^2)}\left(\frac{0}{0}\right)\overset{L'H}{=}\lim_{x\to 0^+}\frac{\sin x}{\frac{2x}{1+x^2}}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\left(\frac{1+x^2}{2}\right)=1\cdot\frac{1}{2}=\frac{1}{2}$$

Since $\lim_{x\to 0^+} f(x)\neq\lim_{x\to 0^-} f(x)$, $f(x)$ can't be continuous at $x=0$.

$\therefore$ There's no c which makes $f(x)$ continuous at $x=0$

Question 2. (9+8+8 pts) This problem has three unrelated parts about integrals.

(a) Evaluate $\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx$.

$\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty$. Hence, this integral is improper.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx = \lim_{c\to 0^+} \int_c^1 \frac{\ln x}{\sqrt{x}}\, dx = \lim_{c\to 0^+} \left[2\sqrt{x}\ln x \Big|_c^1 - 2\int_c^1 \frac{1}{\sqrt{x}}\, dx \right]$$

$u = \ln x \quad du = \frac{1}{x}dx$

$dv = \frac{1}{\sqrt{x}}dx \quad v = 2\sqrt{x}$

$$= \lim_{c\to 0^+} \left[\left(2\sqrt{x}\ln x - 4\sqrt{x}\right)\Big|_c^1 \right] = \lim_{c\to 0^+} \left[(2\ln 1 - 4) - (2\sqrt{c}\ln(c) - 4\sqrt{c}) \right] = -4$$

$$\lim_{c\to 0^+} \sqrt{c}\ln(c)\ (0\cdot\infty) = \lim_{c\to 0^+} \frac{\ln(c)}{\frac{1}{\sqrt{c}}} \left(\frac{-\infty}{\infty}\right) \overset{L'H}{=} \lim_{c\to 0^+} \frac{\frac{1}{c}}{-\frac{1}{2}\cdot\frac{1}{c^{3/2}}} = \lim_{c\to 0^+} -2\cdot\sqrt{c} = 0$$

(b) Does the improper integral $\int_0^\infty \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx$ converge or diverge? **Explain.**

$$\int_0^\infty \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx = \int_0^1 \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx + \int_1^\infty \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx$$

$\frac{2+e^{-x}}{\sqrt[3]{x}} > 0$ on $(0,+\infty)$

$\frac{2+e^{-x}}{\sqrt[3]{x}} \leqslant \frac{3}{\sqrt[3]{x}}$ on $(0,1)$.

$$\underbrace{\int_0^1 \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx}_{\text{convergent by Comp. Thm.}} \leq \underbrace{3\int_0^1 \frac{1}{x^{1/3}}\, dx}_{\text{convergent by p-test}}$$

$\frac{2+e^{-x}}{\sqrt[3]{x}} > \frac{1}{\sqrt[3]{x}}$ on $(1,\infty)$

$$\underbrace{\int_1^\infty \frac{1}{x^{1/3}}\, dx}_{\text{divergent by p-test}} < \underbrace{\int_1^\infty \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx}_{\text{divergent by comp. thm}}$$

$\therefore \int_0^\infty \frac{2+e^{-x}}{\sqrt[3]{x}}\, dx$ is divergent

(c) Evaluate $\int_0^1 \frac{x^2}{(\sqrt{4-x^2})^3}\, dx$.

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\int_0^1 \frac{x^2\, dx}{(\sqrt{4-x^2})^3} = \int_0^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{(\sqrt{4-4\sin^2\theta})^3} = \int_0^{\pi/6} \frac{8\cdot\sin^2\theta\cos\theta\, d\theta}{8\cdot\cos^3\theta}$$

$$= \int_0^{\pi/6} \tan^2\theta\, d\theta = \int_0^{\pi/6} (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta \Big|_0^{\pi/6}$$

$$= \left(\tan\tfrac{\pi}{6} - \tfrac{\pi}{6}\right) - (\tan 0 - 0)$$

$$= \frac{\sqrt{3}}{3} - \frac{\pi}{6}.$$